Introduction to Differential Equations and Applications

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These are the course notes that I used as an instructor of AMATH 351: Introduction to Differential Equations and Applications for Summer Quarter 2023. For roughly the first half of the course, I primarily used Bernard Deconinck's course notes (found here). For the remainder of the course, I used my own course notes, which were themselves adapted from *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition) and the course notes of Ami Radunskaya (Pomona College). All original notes are contained in this document, and all external sources are linked to in the table of contents. Plots included in these notes were generated using a combination of Desmos, Python, and TikZ.

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Lecture 7: Existence and Uniqueness for First Order ODEs

These lecture notes were adapted from Chapter 2.4 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

7.1 Introduction

So far in this course, we've learned various methods for solving different types of first order ODEs (linear, separable, exact, etc.), and we've seen a few applications of such equations in the natural sciences. We also just learned how to analyze the long time behavior of solutions to first order *autonomous* ODEs without solving said equations via phase line analysis.

In teaching these methods, I've made several asides about assumptions on where solutions are "valid" when it comes to specifying the domain and range of these functions. However, we have not yet seen a proper treatment of how to determine for a given IVP which values of t and y solutions of the form $y(t)$ do or do not exist.

We have also seen examples of *nonlinear* equations with multiple solutions (e.g., separable equations with a constant solution other than the solution we find via integration, our so called "divide by zero" cases). Thinking back to the first lecture and our parallels between algebraic equations and differential equations, nonlinear algebraic equations often have multiple solutions, it should not surprise us that the same is also true for differential equations.

As such, we will be approaching the problems of "When does a solution exist?" and "Is there more than one solution?" for first order problems by comparing properties of linear vs. nonlinear ODEs.

7.2 Existence and Uniqueness for First Order Linear ODEs

We begin with a theorem.

Existence and Uniqueness Theorem (First Order Linear). Given the IVP

$$y' + p(t)y = q(t), \quad y(t_0) = y_0, \tag{7.1}$$

suppose the functions p and q are continuous on some open interval $\alpha < t < \beta$ that contains the point $t = t_0$. Then there exists a unique function $y = \phi(t)$ that satisfies the initial condition in (7.1) and solves the differential equation in (7.1) for every t in the interval (α, β).

This theorem states that an IVP of the form (7.1) not only *has* a solution, but has *exactly one* solution so long as the functions p and q are continuous in some interval about the point $t = t_0$. I won't give a proof of this Theorem in class, but Boyce & DiPrima §2.1 and §2.4 give an outline of said proof using the method of integrating factors.

Instead, I will give some illuminating examples of how to apply this Theorem to an IVP to determine the intervals on which the problem has a unique solution.

Example 7.1. Consider the IVP

$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases} \tag{7.2}$$

We note that the ODE is linear, so we can apply the Theorem of Existence and Uniqueness to find an interval on which this problem has a unique solution. Not only will we find *an* interval, we will find the largest possible interval. We begin by putting the ODE into standard form by dividing both sides of the equation by t:

$$y' + \frac{2}{t}y = 4t\ .$$

Identifying p and q:

$$p(t) = \frac{2}{t}, \quad q(t) = 4t\,.$$

We see that $p(t)$ is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$, i.e., it is continuous everywhere but $t = 0$. A bit more rigorously, we can look at the left- and right-hand limits of $p(t)$ as t approaches 0 and see that they do not match:

$$\lim_{t \to 0^-} \frac{2}{t} = -\infty, \quad \lim_{t \to 0^+} \frac{2}{t} = \infty.$$

The function $q(t)$ is continuous on all intervals of t.

In order to apply our Theorem, we need to find an interval on which *both* p and q are continuous. By what we just deduced, the only two such intervals are $t < 0$ and $t > 0$. Looking at the Theorem, the interval on which a unique solution to our IVP exists must contain the point $t = t_0$. For the problem (7.2), this point is $t_0 = 1$, which lies in the interval $t > 0$.

Therefore, a unique solution to this IVP exists on the interval $t > 0$. I won't go through the steps here, but the solution to this initial value problem is

$$y(t) = t^2 + \frac{1}{t^2}\,.$$

(Feel free to check this yourself using the method of integrating factors). Now that we know to be more careful when it comes to specifying the domain on which this function solves the given IVP, we should really write this statement as

$$y(t) = t^2 + \frac{1}{t^2}\,, \quad t > 0.$$

Example 7.2. Suppose we have the same ODE as in (7.2), but instead have initial condition $y(-1) = 2$. Then using what we know about the continuity of p and q and applying the Theorem of Existence and Uniqueness, we would conclude that this new IVP has a unique solution for $t < 0$. In fact, its solution is

$$y(t) = t^2 + \frac{1}{t^2}\,, \quad t < 0.$$

(Verify on your own).

7.3 Existence and Uniqueness for All First Order ODEs

There is also a Theorem of Existence and Uniqueness that applies more broadly to all first order ODEs, both linear and nonlinear. It is as follows.

Existence and Uniqueness Theorem (First Order, General). Given the IVP

$$y' = f(t, y), \quad y(t_0) = y_0, \tag{7.3}$$

suppose the functions f and $\partial f/\partial y$ are continuous on some open rectangle $\alpha < t < \beta$, $a < y < b$ that contains the point $(t, y) = (t_0, y_0)$. Then there exists a unique function $y = \phi(t)$ that satisfies the initial condition in (7.3) and solves the differential equation in (7.3) for every t in some sub-interval $(t_0 - h, t_0 + h)$ of (α, β).

We observe that the hypotheses on f and $\partial f/\partial y$ reduce to those on p and q in the previous theorem if the given ODE is linear. In that case, we would have

$$f(t, y) = -p(t)y + q(t) \implies \frac{\partial f}{\partial y} = -p(t)\,,$$

so that continuity of p and q is equivalent to continuity of f and $\partial f/\partial y$.

Remark: Continuity of f alone guarantees existence of solutions, but not uniqueness.

As before, I will not give a proof of this Theorem nor the above remark, as is is beyond the scope of this course, but I will point you to Boyce & DiPrima §2.8 for further reading. Similar to the linear case, we will go through an example of how to apply this Theorem to an IVP.

Example 7.3. Consider the IVP

$$\begin{cases} y' = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1 \end{cases} \tag{7.4}$$

We saw this IVP in an earlier lecture, when we were learning about separation of variables. In order to apply our Theorem, we observe:

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}.$$

We will take as a fact that the quotient of two continuous functions is also a continuous function except where its denominator is equal to zero. Then each of these functions is continuous at every point (x, y) except along the line $y = 1$.

We certainly can draw a rectangle about the initial point $(x_0, y_0) = (0, -1)$ that does not include the line $y = 1$, e.g., $-5 < x < 5$, $-2 < y < 0$. Both f and $\partial f/\partial y$ will be continuous on such a rectangle. In fact, we could extend the rectangle in the direction of x infinitely in both directions and still have continuity of these functions. However, the Theorem of Existence and Uniqueness for nonlinear problems only guarantees the existence of a unique solution on an *sub*-interval of the interval on x that we used to define our rectangle. In fact, when we solved this IVP in class, we found an explicit solution (Figure 1):

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{7.5}$$

This is one general difference between linear and nonlinear problems: we know that we can always find an *explicit solution* to a linear ODE (see the method of integrating factors), but oftentimes the best we can do for nonlinear problems is an *implicit solution* (see exact ODEs).

Back to determining an interval for existence and uniqueness for problem (7.4), we note that saw that the solution (7.5) only exists for x greater than the value of x such that $y(x) = 1$, which ends up being $x = 2$. That is, the IVP (7.5) has a unique solution on the interval $x > 2$. This another general difference between linear and nonlinear problems: it is often more difficult to determine an interval on which a unique solution exists when dealing with a nonlinear problem.

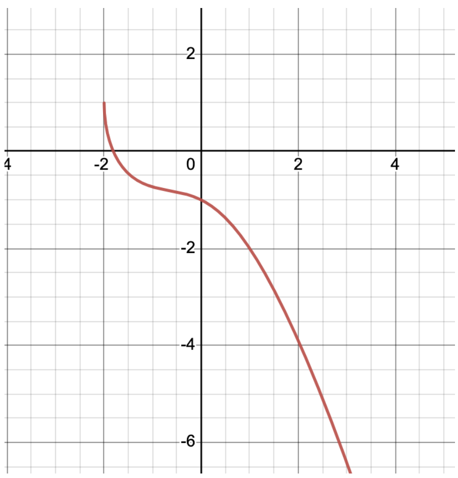

Figure 1: Plot of the explicit solution (7.5) of the IVP (7.4).

Example 7.4. Suppose we have the same ODE as in (7.4), but instead have initial condition $y(0) = 1$. In this case, the initial point (x_0, y_0) lies on the line $y = 1$, so no rectangle can be drawn around this point such that f and $\partial f/\partial y$ are continuous. Thus, we cannot apply the Theorem of Existence and Uniqueness to say anything about possible solutions to this problem.

If we were to separate variables and solve the IVP, we would get the implicit solution

$$y^2 - 2y = x^3 + 2x^2 + 2x - 1\,.$$

In this case, when we complete the square and find the explicit solution, there is not a clear choice of the + versus the − square root, as both functions satisfy the initial condition $y(0) = 1$:

$$y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x}\,. \tag{7.6}$$

Both of these functions certainly satisfy the ODE, and they both satisfy the initial condition, which means they both solve the IVP. Therefore, there are two distinct solutions to this IVP; we have existence, but no uniqueness.

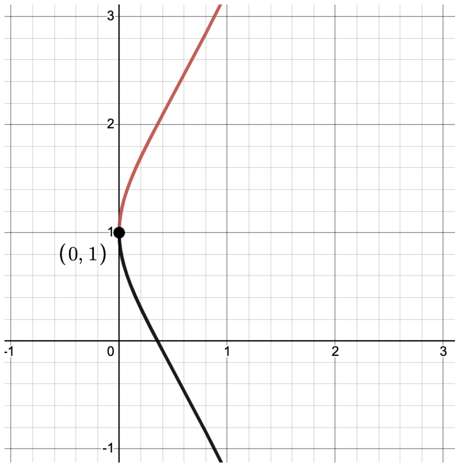

Figure 2: Plot of the two explicit solutions (7.6) of the ODE in (7.4) with initial conditions $y(0) = 1$.

Supplemental Notes for Lecture 9: Existence and Uniqueness for Second Order ODEs

These lecture notes were adapted from Chapter 3.2 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

In Lecture 9, we considered more generally 2nd order equations of the form

$$y'' + p(x)y' + q(x)y = 0\,, \tag{1}$$

which are *second order, linear, homogeneous* ODEs with *non-constant coefficients*. We will see more of such ODEs in future lectures, but in Lecture 9, we considered these ODEs in the context of Abel's theorem. We applied this theorem in the *reduction of order method* to solve the constant coefficient equation with repeated roots, but the theorem holds more generally for equations of the form (1).

In particular, Abel's theorem requires that we have two solutions to a given ODE (1). How and when do we know that such solutions exist? As it turns out, much like the case with linear first order equations, there is a **Existence and Uniqueness Theorem** for linear second order equations.

Existence and Uniqueness Theorem (Second Order Linear). Given the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = v_0, \tag{2}$$

suppose the functions p, q and g are continuous on some open interval $\alpha < t < \beta$ that contains the point $t = t_0$. Then there exists a unique function $y = \phi(t)$ that satisfies the initial conditions in (2) and solves the differential equation in (2) for every t in the interval (α, β).

We note that, so far, we have only seen second order homogeneous equations, i.e., $g(t) = 0$ in (2). Therefore, when applying the theorem of existence and uniqueness to an equation like (1) with some initial conditions, we need only consider the interval on which p and q are both continuous.

Example. Consider the IVP

$$\begin{cases} (t^2 - 3t)y'' + ty' - (t+3)y = 0, \\ y(1) = 2, \quad sy'(1) = 1, \end{cases} \tag{3}$$

We note that the ODE is a second order, linear, homogeneous equation, so we can apply the Existence and Uniqueness Theorem to find an interval on which this problem has a unique solution. Not only will we find *an* interval, we will find the largest possible interval. We begin by putting the ODE into standard form by dividing both sides of the equation by t:

$$y'' + \frac{1}{t-3}y' - \frac{t+3}{t(t-3)}y = 0\ .$$

Identifying p and q:

$$p(t) = \frac{1}{t-3}, \quad q(t) = -\frac{t+3}{t(t-3)}\,.$$

We see that $p(t)$ is continuous on the intervals $(-\infty, 3)$ and $(3, \infty)$, i.e., it is continuous everywhere but $t = 3$. Similarly, the function $q(t)$ is not continuous at $t = 0$ and $t = 3$, but is continuous on the intervals $(-\infty, 0)$, $(0, 3)$, and $(3, \infty)$.

In order to apply our Theorem, we need to find an interval on which *both* p and q are continuous. By what we just deduced, the only such intervals are $(-\infty, 0)$, $(0, 3)$, and $(3, \infty)$. Looking at the Theorem, the interval on which a unique solution to our IVP exists must contain the point $t = t_0$. For the problem (3), this point is $t_0 = 1$, which lies in the interval $0 < t < 3$. Therefore, a unique solution to this IVP exists on the interval $0 < t < 3$.

Lecture 14: Introduction to the Laplace Transform

These lecture notes were adapted from Chapter 6.1 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

14.1 Introduction

Last lecture, we learned about the method of variation of parameters, which allowed as to solve

$$y'' + p(x)y' + q(x)y = g(x) \tag{14.1}$$

for any given $g(x)$. Consider, however, an ODE of the form (14.1) with $g(x)$ defined as

$$g(x) = \begin{cases} 0 & 0 \leqslant x < 5\,, \\ 1 & 5 \leqslant x < 20\,, \\ 0 & x \geqslant 20\,. \end{cases} \tag{14.2}$$

A plot of this function is given in Figure 3.

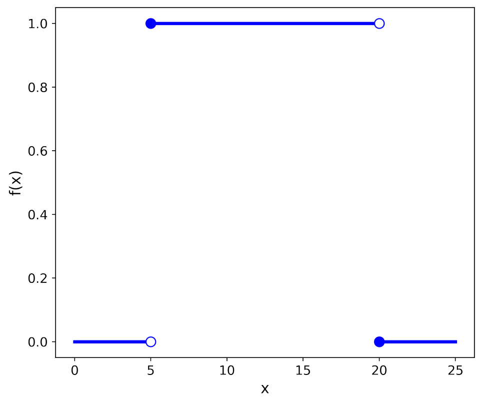

Figure 3: Plot of the discontinuous function $g(x)$ as given in (14.2).

This non-homogeneous term is certainly discontinuous; in particular, we say that this function is *piecewise continuous*, a term that we will define and discuss later. These types of ODEs often show up in engineering applications for mechanical or electrical systems, among other applications.

We could use one of our previous methods, such as variation of parameters, to solve a second order ODE with non-homogeneous term of this form. However, it would require us to do even more work when it comes to dealing with the points at which $g(x)$ is discontinuous.

Instead, over the next few lectures, we will work towards learning how to solve linear ODEs with discontinuous forcing functions using something called a *Laplace transform*. This method is something that is useful more broadly in the context of linear ODEs, but is particularly useful for ones with discontinuous forcing functions. In this lecture, we will define the Laplace transform and learn about some of its important properties.

For a function $f(t)$ defined on $t \geqslant 0$, we define the Laplace transform of f as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt\,,$$

where s is a parameter that could take on real or complex values. In this class, we will only focus on s as a real number.

The Laplace transform belongs to a more general class of relations called *integral transforms*, which generally take the form:

$$F(s) = \int_{\alpha}^{\beta} K(s,t)f(t)\,dt\,,$$

where $K(s,t)$ is a given function, called the *kernel* of the transformation. Integral transforms are a widely used tool in the solution of differential equations, both ordinary and partial.

14.2 Improper Integrals and Piecewise Continuity

Because the Laplace transform involves an integral from zero to infinity, we should review **improper integrals** of this type:

$$\int_{a}^{\infty} f(t)dt\,.$$

Recall that an improper integral over an *unbounded* interval is defined as the *limit* of integrals over finite integrals:

$$\int_{a}^{\infty} f(t)dt = \lim_{b\to\infty}\int_{a}^{b} f(t)dt\,.$$

If both the integral from a to b for every $b > a$ *and* the limit as $b \to \infty$ exists, then we say that the above integral **converges** to that value. Else, we say that the improper integral **diverges**.

Example 14.1. Let $f(t) = e^{ct}$ where $t \geqslant 0$, and $c \neq 0$ is a real constant. Then

$$\int_{0}^{\infty} e^{ct}dt = \lim_{b\to\infty}\int_{0}^{b} e^{ct}dt = \lim_{b\to\infty}\left[\frac{e^{ct}}{c}\right]_0^b = \lim_{b\to\infty}\frac{1}{c}\left(e^{cb}-1\right).$$

If $c < 0$, then

$$\lim_{b\to\infty}\frac{1}{c}\left(e^{cb}-1\right) = \frac{1}{c}(0-1) = -\frac{1}{c}\,.$$

Therefore, the improper integral converges when $c < 0$. However, if $c > 0$, then

$$\lim_{b\to\infty}\frac{1}{c}e^{cb} = +\infty\,.$$

Thus, the improper integral diverges when $c < 0$. We note that when $c = 0$, $f(t) = 1$, and the integral also diverges.

Example 14.2. Let $f(t) = t^{-p}$, where $t \geqslant 1$ and p is a real number. We will consider the following improper integral of f

$$\int_{1}^{\infty} t^{-p}dt = \lim_{b\to\infty}\int_{1}^{b} t^{-p}dt\,.$$

for $p \neq 1$ and $p = 1$. First, let $p = 1$; then we have:

$$\lim_{b\to\infty}\int_{1}^{b}\frac{1}{t}dt = \lim_{b\to\infty}\ln(b) = +\infty\,.$$

Therefore, the improper integral diverges for $p = 1$. Now, let $p \neq 1$; then we have:

$$lim_{b\to\infty}\int_{1}^{b} t^{-p}dt = \lim_{b\to\infty}\frac{1}{1-p}\left(b^{1-p}-1\right).$$

If $p > 1$, this quantity converges to the value 0; if $p < 1$, this value diverges to $+\infty$. In sum, the improper integral of t^{-p} from 1 to ∞ converges of $p > 1$, and diverges if $p \leqslant 1$.

Note. This result is analogous to that for the infinite series $\sum\limits_{n=1}^{\infty} n^{-p}$.

In general, to talk about whether or not an improper integral of a function f over an unbounded integral exists (i.e., converges or diverges), it will be helpful for us to define a few concepts.

Definition. A function $f(t)$ is **piecewise continuous** on an interval $\alpha \leqslant t \leqslant \beta$ if there are a finite number of points $\alpha < t_0 < t_1 < \cdots < t_n < \beta$ such that

1. $f(t)$ is continuous on each (open, sub-) interval $t_{i-1} < t < t_i \quad (i = 1, 2, \ldots n)$;
2. $f(t)$ approaches a finite limit as the endpoints t_{i-1} and t_i of each sub-interval are approached from within that interval.

If $f(t)$ is piecewise continuous on all intervals $\alpha \leqslant t \leqslant \beta$ with $\beta > \alpha$, we say that it is piecewise continuous on the unbounded interval $t \geqslant \alpha$.

Remark. In other words, a function f is piecewise continuous on an interval $\alpha \leqslant t \leqslant \beta$ if it is continuous on that interval except for a finite number of jump discontinuities.

Example 14.3. The function $g(x)$ as defined in (14.2) is piecewise continuous on $x \geqslant 0$.

Example 14.4. The function

$$f(x) = \begin{cases} x - 1 & 0 \leqslant x < 4\,, \\ \dfrac{(x-8)^2}{4} & 4 \leqslant x < 12\,, \\ 1 & x \geqslant 12\,. \end{cases} \tag{14.3}$$

is piecewise continuous on $x \geqslant 0$ (Figure 4 a).

Example 14.5. The function

$$f(x) = \begin{cases} \frac{1}{x-1} & 0 \leqslant x < 1\,, \\ 2 & 1 \leqslant x < 5\,, \\ x + 2 & x \geqslant 5\,. \end{cases} \tag{14.4}$$

is *not* piecewise continuous on $x \geqslant 0$, because there is a vertical asymptote when approaching the point $x = 1$ from the left (Figure 4 b). However, it is piecewise continuous on $x \geqslant 1$.

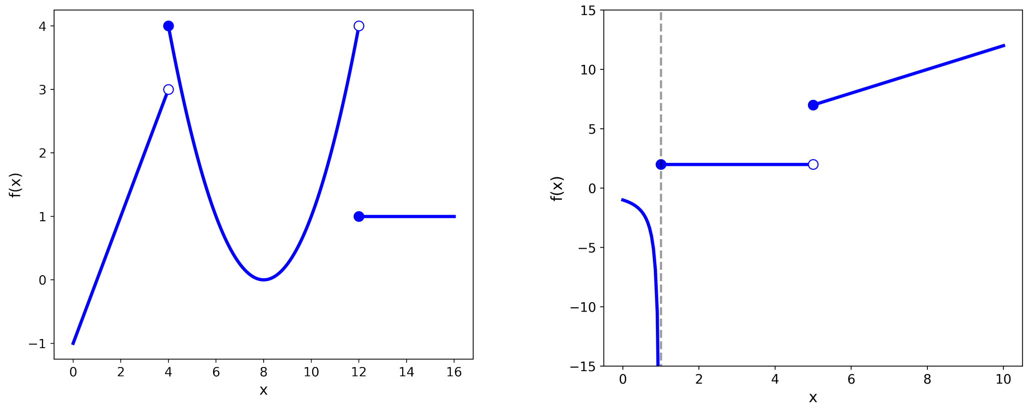

Figure 4: Plot of the functions $f(x)$ as given in (a) Eq. (14.3) and (b) Eq. (14.4).

As it turns out, the integral of a piecewise continuous function over a finite interval is equal to the sum of the integrals of said function over the sub-intervals on which it is continuous:

$$\int_\alpha^\beta f(t)dt = \int_\alpha^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \cdots + \int_{t_n}^{\beta} f(t)dt\,.$$

Thinking back to our discussion of improper integrals, if $f(t)$ is piecewise continuous on the unbounded interval $t \geqslant a$, then the integral

$$\int_a^b f(t)dt$$

exists for every $b > a$. Is this enough for us to say that

$$\int_a^\infty f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt$$

converges? Unfortunately, no. Fortunately, we do have the following theorem that tells us when the integral of a piecewise function over an unbounded interval converges (in the form of a comparison test, similar to that for infinite series).

Theorem. Suppose $f(t)$ is piecewise continuous for $t \geqslant a$. Further suppose that there is some function $g(t)$ and positive constant M such that

i) $|f(t)| \leqslant g(t)$ for $t \geqslant M$,

ii) $\int_M^\infty g(t)dt$ converges,

then the integral $\int_a^\infty f(t)dt$ converges. On the other hand, if

i) $f(t) \geqslant g(t) \geqslant 0$ for $t \geqslant M$,

ii) $\int_M^\infty g(t)dt$ diverges,

then the integral $\int_a^\infty f(t)dt$ diverges.

We will not give the proof here. The functions $g(t) = e^{ct}$ and $g(t) = t^{-p}$ (which we covered in Examples 1 & 2) are common choices for this comparison test. In particular, functions that are bounded by e^{ct} are a special class of functions for which we have a name.

Definition. A function f is of **exponential order** a for some real constant a if there exist $K > 0$ and $M > 0$ such that $|f(t)| \leqslant Ke^{at}$ when $t \geqslant M$.

14.3 The Laplace Transform

We now have everything we need to define and discuss the Laplace transform.

Definition. Let $f(t)$ be a function defined on $t \geqslant 0$. The **Laplace transform** of f, denoted $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

over values of s such that this improper integral converges.

We will discuss how we determine the values of s such that the integral $F(s) = \mathcal{L}\{f\}$ converges in a bit. First, some simple examples.

Example 14.6. Let $f(t) = 1$, where $t \geqslant 0$. Then applying our result from Example 14.1,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s > 0\,.$$

If $s \leqslant 0$, then the above integral diverges, so the Laplace transform does not exist.

Example 14.7. Let $f(t) = e^{at}$, where $t \geqslant 0$. Then again applying our result from Example 14.1,

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{-st}e^{-at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{s-a}, \quad s > a\,.$$

If $s \leqslant a$, then the above integral diverges, so the Laplace transform does not exist.

The following theorem does give us a *sufficient* condition on $f(t)$ for $F(s)$ to exist.

Theorem. Let $f(t)$ be a function that is piecewise continuous on $t \geqslant 0$ and is of exponential order a for some real constant a. Then the Laplace transform of f, $\mathcal{L}\{f\} = F(s)$, exists for $s > a$.

Remark: We note that this theorem provides *sufficient* conditions for $F(s)$ to exist over $s > a$, but not *necessary* conditions. That is, we could have a function $f(t)$ that is of exponential order a, but whose Laplace transform exists for $s > \alpha$ where $\alpha < a$. For example, you will show on this homework that the Laplace transform of $f(t) = t^n$, where n is a positive integer, exists for $s > 0$. However, t^n is not of exponential order zero, as exponential order zero means that for some $K > 0$ and $M > 0$, $|f(t)| \leqslant K$ for $t \geqslant M$. In other words, exponential order zero means that our function is *bounded* by a constant past a certain value of t; this is certainly is not true for t^n for $n \geqslant 1$. Therefore, in practice, we sometimes determine the range of s for which $F(s)$ exists in the context of the given problem (usually by using the comparison test and/or certain known limits).

Example 14.8. Let $f(t) = \sin(at)$, where $t \geqslant 0$. We note that $f(t)$ is piecewise continuous and of exponential order zero, as $|\sin(at)| \leqslant 1$ for all t. Then

$$\mathcal{L}\{\sin(at)\} = F(s) = \int_0^\infty e^{-st}\sin(at)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\sin(at)\,dt \quad s > 0\,.$$

We will use integration by parts to evaluate this integral, taking $u = e^{-st}$ and $v = -(1/a)\cos(at)$, so that $du = -se^{-st}dt$ and $dv = \sin(at)dt$. We have:

$$\begin{aligned}
F(s) &= \lim_{b\to\infty}\int_0^b e^{-st}\sin(at)\,dt = \lim_{b\to\infty}\int_0^b u\,dv \\
&= \lim_{b\to\infty}\left(uv\Big|_0^b - \int_0^b v\,du\right) \\
&= \lim_{b\to\infty}\left[-\frac{e^{-st}\cos(at)}{a}\Bigg|_{t=0}^{t=b} - \frac{s}{a}\int_0^b e^{-st}\cos(at)\,dt\right] \\
&= \lim_{b\to\infty} -\frac{e^{-sb}\cos(ab)}{a} + \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,dt \\
&= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,dt
\end{aligned}$$

Performing integration by parts again on the integral $\int_0^\infty e^{-st}\cos(at)\,dt$ would give us:

$$\int_0^\infty e^{-st}\cos(at)\,dt = \frac{s}{a}\int_0^\infty e^{-st}\sin(at)\,dt = \frac{s}{a}F(s)$$

Plugging this back into our expression for $F(s)$, we have:

$$F(s) = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,dt = \frac{1}{a} - \frac{s^2}{a^2}F(s)\,.$$

Solving for $F(s)$, we get:

$$\boxed{F(s) = \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}\,, \quad s > 0}\,.$$

We note that in the second integration by parts, we found that

$$\mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,dt = \frac{s}{a}\mathcal{L}\{\sin(at)\}\,.$$

Therefore, the Laplace transform of $\cos(at)$ is

$$\boxed{\mathcal{L}\left\{\cos(at)\right\} = \frac{s}{s^2+a^2}\,, \quad s>0}\,.$$

Example 14.9. Let $f(t) = 5e^{-2t} - 3\sin(4t)$, where $t \geqslant 0$. We now know how to find $\mathcal{L}\left\{e^{-2t}\right\}$ and $\mathcal{L}\left\{\sin(4t)\right\}$. How do we find $\mathcal{L}\left\{f\right\}$ in this case?

Theorem. Let f_1 and f_2 be two functions defined on $t \geqslant 0$ whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then the Laplace transform of the linear combination $c_1 f_1 + c_2 f_2$ is defined for $s > \max(a_1, a_2)$, with

$$\mathcal{L}\left\{c_1 f_1 + c_2 f_2\right\} = c_1 \mathcal{L}\left\{f_1\right\} + c_2 \mathcal{L}\left\{f_2\right\}\,.$$

We say that the Laplace transform is a **linear operator**.

Proof. Using linearity of the integral, we have:

$$\begin{aligned}
\mathcal{L}\left\{c_1 f_1 + c_2 f_2\right\} &= \int_0^\infty e^{-st}\Big[c_1 f_1(t) + c_2 f_2(t)\Big]\,dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt \\
&= c_1 \mathcal{L}\left\{f_1\right\} + c_2 \mathcal{L}\left\{f_2\right\}\,.
\end{aligned}$$

The above equation tells us that the largest interval on which $\mathcal{L}\left\{c_1 f_1 + c_2 f_2\right\}$ exists is the largest interval on which both $\mathcal{L}\left\{f_1\right\}$ and $\mathcal{L}\left\{f_2\right\}$ exist; this interval is $s > \max(a_1, a_2)$. ■ ■

Example 14.10. Let $f(t) = 5e^{-2t} - 3\sin(4t)$, where $t \geqslant 0$. Then we can apply this theorem to get:

$$F(s) = \mathcal{L}\left\{f\right\} = 5\mathcal{L}\left\{e^{-2t}\right\} - 3\mathcal{L}\left\{\sin(4t)\right\} = \frac{5}{s+2} - \frac{12}{s^2+16}\,,$$

where $F(s)$ exists for $s > \max(-2, 0) = 0$.

Next lecture, we will see how we can use the Laplace transform to solve IVPs.

Lecture 15: Solving IVPs with the Laplace Transform

These lecture notes were adapted from Chapter 6.2 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

Now that we have defined the Laplace transform of a function and established some of its basic properties (existence/convergence, linearity), we will show how it can be used to solve IVPs for linear ODEs with constant coefficients. The Laplace transform is particularly useful for ODEs of this type because the Laplace transform of the derivatives of a function can be written directly in terms of the original function and its transform. Recall some notation for higher order derivatives:

$$f^{(n)}(x) = \frac{d^n f}{dx^n}(x)\,.$$

Theorem. Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous, and that $f^{(n)}$ is piecewise continuous, on any interval $0 \leqslant t \leqslant b$. Further suppose that $f, f', \ldots, f^{(n-1)}$ are of exponential order a for some constant a. Then $\mathcal{L}\left\{f^{(n)}\right\}$ exists for $s > a$ and is given by:

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

Remark. For $n = 1$, this relation reads:

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)}\,, \tag{15.1}$$

and for $n = 2$, we have:

$$\boxed{\mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)}\,. \tag{15.2}$$

Proof. We will prove this statement for $n = 1$. To start, consider the integral

$$\int_0^b e^{-st} f'(t)\,dt\,.$$

The hypothesis of the theorem states that f' is piecewise continuous on all intervals $0 \leqslant t \leqslant b$. Then f' has at most a finite number of points of discontinuity on an interval $[0, b]$, denote them by $0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant b$; note that m could equal zero. Then we can write the above integral as:

$$\begin{aligned}\int_0^b e^{-st} f'(t)\,dt &= \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \cdots + \int_{t_m}^{b} e^{-st} f'(t)\,dt \\ &= e^{-st}f(t)\Big|_0^{t_1} + e^{-st}f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st}f(t)\Big|_{t_m}^{b} \\ &\qquad + s\left[\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \cdots + \int_{t_m}^{b} e^{-st} f(t)\,dt\right]\end{aligned}$$

where we integrated by parts ($u = e^{-st}$, $v = f(t)$) to obtain the second line. By the theorem hypothesis, f is continuous on $[0, b]$ for any b. Then

$$\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \cdots + \int_{t_m}^{b} e^{-st} f(t)\,dt = \int_0^{b} e^{-st} f(t)\,dt\,.$$

Furthermore,

$$\begin{aligned}
e^{-st}f(t)\Big|_0^{t_1} + e^{-st}f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st}f(t)\Big|_{t_m}^{b} &= e^{-st_1}f(t_1) - f(0) + e^{-st_2}f(t_2) - e^{-st_1}f(t_1) \\
&\quad + \cdots + e^{-sb}f(b) - e^{-st_m}f(t_m) \\
&= e^{-sb}f(b) - f(0)
\end{aligned}$$

As all of the middle terms with $t_1, t_2 \ldots, t_m$ cancel out (this type of sum is called a *telescoping sum*). Combining these results, we get:

$$\int_0^b e^{-st}f'(t)\,dt = e^{-sb}f(b) - f(0) + s\int_0^b e^{-st}f(t)\,dt\,.$$

Taking the limit of both sides as $b \to \infty$:

$$\mathcal{L}\left\{f'(t)\right\} = \lim_{b\to\infty}\left[e^{-sb}f(b)\right] - f(0) + s\mathcal{L}\left\{f(t)\right\}\,.$$

The theorem hypothesis states that f and f' are is of exponential order a for some a. Then $\mathcal{L}\{f'\}$ and $\mathcal{L}\{f\}$ are guaranteed to exist for $s > a$. Furthermore, it follows that

$$\lim_{b\to\infty} e^{-sb}f(b) = 0 \quad \text{when } s > a.$$

Therefore, we have shown that for $s > a$,

$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)\,.$$

■

Recall our general second order linear ODE with constant coefficients:

$$ay'' + by' + cy = g(t)\,. \tag{15.3}$$

We have seen several methods to solve this type of equation (undetermined coefficients, variation of parameters). Now, we will show how to solve this equation using the Theorem that we just learned.

Let $y(t)$ denote the solution to the ODE (15.3) with some initial conditions $y(0)$ and $y'(0)$. Let $Y(s) = \mathcal{L}\{y\}$, and let $G(s) = \mathcal{L}\{g\}$. Taking the Laplace transform of both sides of the ODE (15.3) and using the fact that the Laplace transform is linear, we get:

$$a\mathcal{L}\left\{y''\right\} + b\mathcal{L}\left\{y'\right\} + cY(s) = G(s)\,. \tag{15.4}$$

Applying our theorem, we know that:

$$\mathcal{L}\left\{y''\right\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\left\{y'\right\} = sY(s) - y(0)\,.$$

Substituting these expressions into (15.4) gives us:

$$a\Big[s^2Y(s) - sy(0) - y'(0)\Big] + b\Big[sY(s) - y(0)\Big] + cY(s) = G(s)\,.$$

Finally, we can solve for $Y(s) = \mathcal{L}\{y\}$ to get:

$$\boxed{Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}}\,. \tag{15.5}$$

Therefore, the Laplace transform reduced the problem of solving our non-homogeneous second order ODE to solving an algebraic equation for $Y(s)$. All we would need to do to solve (15.3) under some initial conditions $y(0)$ and $y'(0)$ is to compute $Y(s)$ according to (15.5) and find the function $y(t)$ such that $Y(s) = \mathcal{L}\{y\}$.

Remark. This method for solving linear non-homogeneous ODEs has some "nice" properties:

- The Laplace transform (15.5) of the solution is given directly in terms of the initial conditions $y(0)$ and $y'(0)$, so we would not need to find the general solution and then determine the appropriate values of the arbitrary constants. Note that if we were given initial values of y and y' at some point $t_0 \neq 0$, we could use the Laplace transform solve for the function $u(t) = y(t - t_0)$, and then obtain y via $y(t) = u(t + t_0)$.
- Homogeneous equations and non-homogeneous equations are handled in the exact same way for this method, using the fact that $\mathcal{L}\{0\} = 0$. Unlike our previous methods, we do not have to solve the homogeneous equation first, and instead can directly solve the non-homogeneous problem.
- The general Laplace transform method is not limited to second-order ODEs but can be applied to any order of ODE, so long as it is linear and has constant coefficients.

Returning to our process of obtaining $y(t)$ from $Y(s)$, doing so is called finding the *inverse Laplace transform* of $Y(s)$, denoted $\mathcal{L}^{-1}\{Y(s)\}$. There is a general formula for the inverse Laplace transform of a function, but it involves treating the Laplace transform as a function of a *complex* variable s. For the purposes of this class, we will establish some common Laplace transform/inverse pairs and some important properties of the inverse Laplace transform without using complex variables.

In particular, for the functions $f(t)$ that we will be dealing with in this course (continuous and piecewise continuous functions), we can say that there is essentially a "one-to-one correspondence" between Laplace transforms and their inverses. That is, **if we have two functions f and g such that $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then we can conclude that $f = g$.**[1] There is a **table of common Laplace transform/inverse pairs at the end of these notes** (Table 15.1). We have derived some of these already in class, will derive some in lecture, and will derive others in the homework. Regardless of if/when we derive these pairs, you may take them to be true and use them going forward in this class (i.e., you **do not have to derive them unless explicitly asked to do so**).

Theorem. Let $F_1(s) = \mathcal{L}\{f_1(t)\}$ and $F_2(s) = \mathcal{L}\{f_2(t)\}$ for some functions f_1 and f_2, and define

$$F(s) = c_1 F_1(s) + c_2 F_2(s)$$

for some constants c_1 and c_2. Then

$$\mathcal{L}^{-1}\{F(s)\} = c_1 \mathcal{L}^{-1}\{F_1(S)\} + c_2 \mathcal{L}^{-1}\{F_2(s)\} = c_1 f_1(t) + c_2 f_2(t)$$

We say that the inverse Laplace transform is a **linear operator**.

Note. We will not give the proof here, but a the idea behind the proof is given in Boyce & DiPrima Chapter 6.2. It mainly relies on the uniqueness property of Laplace transforms and their inverses.

Example 15.1. Consider the IVP:

$$y'' + y = \sin(2t), \quad y(0) = 2, \quad y'(0) = 1\,.$$

The coefficients on the left-hand side of this equation are $a = 1, b = 0$, and $c = 1$. In order to use our formula for $Y(s)$, we must find $G(s) = \mathcal{L}\{\sin(2t)\}$. We derived a formula for this Laplace transform

[1] The only small caveat is that two piecewise continuous functions could be equal everywhere except at their points of discontinuity, but still have the same Laplace transform. The reason is because these endpoints do not "contribute" to the value of the integral when we take the Laplace transform.

last lecture!

$$G(s) = \mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}\ ,$$

which exists for $s > 0$. Then $Y(s)$ is given by

$$Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)} \tag{15.6}$$

We know that

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 2\cos(t) + \sin(t)\,,$$

but we don't know how to compute

$$\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+4)}\right\}\ .$$

We do note that if we write this term it as a sum of partial fractions, then it looks a lot more like something we do know how to take the inverse Laplace transform of:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}\ . \tag{15.7}$$

That is, we know that

$$\mathcal{L}^{-1}\left\{\frac{\alpha}{s^2+\alpha^2}\right\} = \sin(\alpha t), \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+\alpha^2}\right\} = \cos(\alpha t)\,,$$

so we want terms that "look like" $\alpha/(s^2+\alpha^2)$ and $s/(s^2+\alpha^2)$ for some real number α.

Writing the right-hand side of (15.7) in terms of the common denominator $(s^2+1)(s^2+4)$ and equating the numerator on both sides, we get:

$$2 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B + D$$

In order for this statement to hold, the coefficients of each power of s on each side of the equation must be equal. This condition gives us a linear algebraic system of equations for A, B, C, and D:

$$\begin{cases} A+C &= 0 \\ B+D &= 0 \\ 4A+C &= 0 \\ 4B+D &= 2 \end{cases}.$$

The solution to these equations is $A = 0$, $B = 2/3$, $C = 0$, $D = -2/3$. Thus,

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{2}{3(s^2+1)} - \frac{2}{3(s^2+4)} = \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\cdot\frac{1}{s^2+(2)^2}\,,$$

which has inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+4)}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\} = \frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)$$

Combining our results, we have that the solution to the given IVP is

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)(s^2+4)}\right\} \\
&= 2\cos(t) + \sin(t) + \frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t) \\
&= \boxed{2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)}.
\end{aligned}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n$ a positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0, \quad \Gamma$ denotes the Gamma function
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at}t^n, \quad n$ a positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}f(t)$	$F(s-a)$
$f(ct), \quad c > 0$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$(-t)^nf(t)$	$F^{(n)}(s)$

Table 15.1: Table of common functions and their Laplace transforms.

Lecture 16: Step Functions and the Laplace Transform

These lecture notes were adapted from Chapter 6.3 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

16.1 Introduction

In the previous lecture, we saw how to use the Laplace transform to solve IVPs for linear ODEs with continuous forcing functions. In this lecture we will establish a few more properties of the Laplace transform that will allow us to solve IVPs for linear ODEs with *discontinuous* forcing functions (particularly those that are piecewise continuous, i.e., only have jump discontinuities).

16.2 Unit Step Function

In order to talk about the Laplace transform of a function with jump discontinuities, it will be useful for us to introduce the following function.

Definition. The **unit step function**, or **Heaviside function**, with constant $c \geqslant 0$ is denoted $u_c(t)$ and defined over $t \geqslant 0$ as

$$u_c(t) = \begin{cases} 0 & 0 \leqslant t < c \\ 1 & t \geqslant c \end{cases}.$$

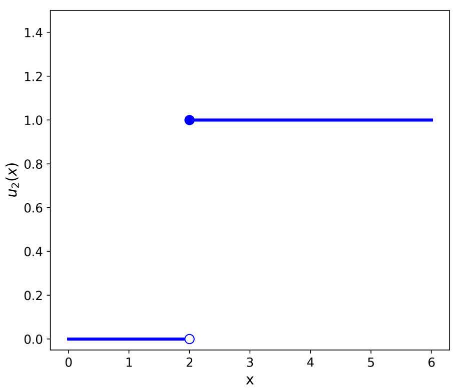

Figure 5: Plot of the unit step function with $c = 2$

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The unit step function is useful to us because it allows us to write piecewise functions in a more compact manner, as seen in the following example.

Example 16.1. Consider the function

$$f(t) = \begin{cases} 2 & 0 \leqslant t < 4 \\ 5 & 4 \leqslant t < 7 \\ -1 & 7 \leqslant t < 9 \\ 1 & t \geqslant 9 \end{cases}.$$

How would we go about writing $f(t)$ in terms of $u_c(t)$ for various values of c? We start by observing that the function $f_1(t) = 2$ is equal to $f(t)$ on the interval $[0, 4)$. In order to get the upward jump of three units at $t = 4$, we add $3u_4(t)$ to f_1:

$$f_2(t) = f_1(t) + 3u_4(t) = 2 + 3u_4(t)\,.$$

Then the function $f_2(t)$ as defined above is equal to $f(t)$ on the interval $[0, 7)$. At $t = 7$, $f(t)$ jumps downward by 6 units; thus, we add $-6u_7(t)$ to f_2 to get a function that is equal to $f(t)$ on $[0, 9)$:

$$f_3(t) = 2 + 3u_4(t) - 6u_7(t)\,.$$

Finally, we add $2u_9(t)$ in order to match the jump upwards of two units at $t = 9$:

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)\,.$$

Example 16.2. Consider the function

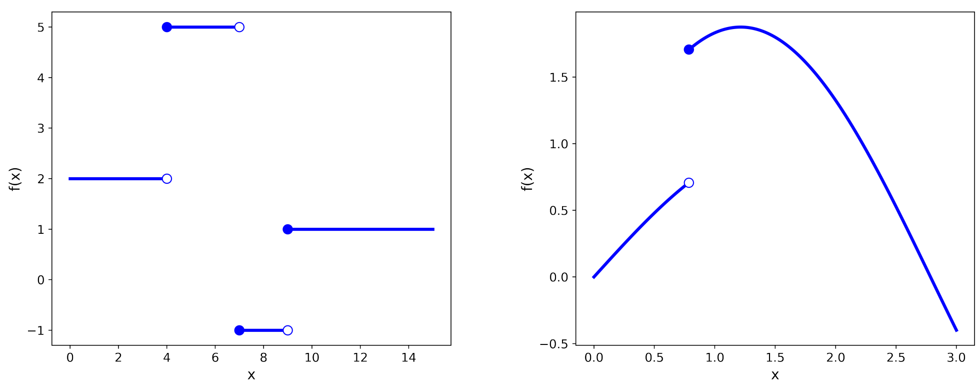
$$f(t) = \begin{cases} \sin(t) & 0 \leqslant t < \dfrac{\pi}{4} \\ \sin(t) + \cos(t - \pi/4) & t \geqslant \dfrac{\pi}{4}\,. \end{cases}$$

It is not immediately clear how we can write this function in terms of u_c. However, we observe that $f(t)$ can be written as $\sin(t) + g(t)$, where

$$g(t) = \begin{cases} 0 & 0 \leqslant t < \dfrac{\pi}{4} \\ \cos(t - \pi/4) & t \geqslant \dfrac{\pi}{4}\,. \end{cases}$$

We can write $g(t) = u_{\pi/4}(t)\cos(t - \pi/4)$, giving us

$$f(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4)\,.$$



(a) (b)

Figure 6: Plot of the functions $f(x)$ as given in (a) Example 16.1 and (b) Example 16.2.

16.3 Laplace Transform and the Unit Step Function

Beyond simplifying the expression for a function with jump discontinuities, the unit step function allows us to "easily" compute the Laplace transform of such a function. For one, the Laplace transform of u_c is straightforward to compute:

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_0^\infty e^{-st}u_c(t)\,dt = \int_0^\infty e^{-st}\cdot 0\,dc + \int_c^\infty e^{-st}\cdot 1\,dt \\ &= \int_c^\infty e^{-st}\,dt = -\frac{1}{s}\lim_{b\to\infty}\left(e^{-sb} - e^{-sc}\right) \\ &= \boxed{\frac{e^{-cs}}{s}, \quad s > 0}\,. \end{aligned}$$

Example 16.3. Returning to f as defined in Example 1, recall that we could write

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$$

Then we have:

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} + 3\mathcal{L}\{u_3(t)\} - 6\mathcal{L}\{u_7(t)\} + 2\mathcal{L}\{u_9(t)\} = \frac{1}{s}\left(2 + 3e^{-3s} - 6e^{-7s} + 2e^{-9s}\right), \quad s > 0.$$

Theorem. Let $f(t)$ be such that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geqslant 0$. Then for any constant $c > 0$,

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for some $F(s)$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\}.$$

Proof. We can compute $\mathcal{L}\{u_c(t)f(t-c)\}$ in a straightforward manner:

$$\begin{aligned}
\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\
&= \int_c^\infty e^{-st}f(t-c)\,dt \\
&= \int_0^\infty e^{-s(\tau+c)}f(\tau)\,d\tau \qquad (\textit{change of variables } \tau = t - c) \\
&= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = e^{-cs}F(s).
\end{aligned}$$

■

Example 16.4. Returning to f as defined in Example 2, recall that we could write

$$f(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4).$$

Then we can apply this theorem to get

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin(t)\} + \mathcal{L}\left\{u_{\pi/4}(t)\cos(t-\pi/4)\right\} \\
&= \mathcal{L}\{\sin(t)\} + e^{-\pi s/4}\mathcal{L}\{\cos(t)\} \\
&= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}, \quad s > 0.
\end{aligned}$$

Example 16.5. Consider the function

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

We will apply this theorem to find its inverse Laplace transform, $\mathcal{L}^{-1}\{F(s)\} = f(t)$. We have:

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}.$$

Recall that

$$\mathcal{L}\{t\} = \frac{1}{s^2} \implies \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Thus,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t - u_2(t)(t-2)\,.$$

We note that this function can also be written as

$$f(t) = \begin{cases} t & 0 \leqslant t < 2 \\ 2 & t \geqslant 2\,. \end{cases}$$

The following theorem establishes an important property of the Laplace transform that is analogous to that established in the previous theorem.

Theorem. Let $f(t)$ be such that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geqslant 0$. Then for any constant c,

$$\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a + c\,.$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$ for some $F(s)$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}$$

Proof. We can compute $\mathcal{L}\left\{e^{ct} f(t)\right\}$ in a straightforward manner:

$$\mathcal{L}\left\{e^{ct} f(t)\right\} = \int_0^\infty e^{-st} e^{ct} f(t)\,dt = \int_0^\infty e^{-(s-c)t} f(t)\,dt = F(s-c)\,.$$

The restriction on the domain $s > a + c$ comes from our Theorem of existence of the Laplace transform: if $f(t)$ is of exponential order a, then $e^{ct} f(t)$ is of exponential order $a + c$. ∎

Example 16.6. Consider the function

$$G(s) = \frac{1}{s^2 - 4s + 5}\,.$$

We will apply this theorem to find $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Completing the square in the denominator of $G(s)$ gives us

$$G(s) = \frac{1}{(s-2)^2 + 1}\,.$$

Then if we define:

$$F(s) = \frac{1}{s^2 + 1}\,,$$

we have that $G(s) = F(s-2)$. Because $\mathcal{L}^{-1}\{F(s)\} = \sin(t)$, it follows that

$$\mathcal{L}^{-1}\{G(s)\} = e^{2t}\sin(t)$$

In the next lecture, we will see several examples of how to use the Laplace transform to solve IVPs whose ODEs have discontinuous forcing functions.

Lecture 17: ODEs with Discontinuous Forcing Functions

These lecture notes were adapted from Chapter 6.4 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

We we will now learn how to solve non-homogeneous linear ODEs whose non-homogeneous term can be written in terms of unit step functions. Let's return to an example that we saw at the beginning of our foray into the Laplace transform.

Example 17.1. Consider the ODE

$$2y'' + y' + 2y = g(t), \tag{17.1}$$

where $g(t)$ is defined as

$$g(t) = \begin{cases} 0 & 0 \leqslant t < 5\,, \\ 1 & 5 \leqslant t < 20\,, \\ 0 & t \geqslant 20\,. \end{cases} \tag{17.2}$$

we will solve this ODE subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$. Using our formula for the Laplace transform $Y(s)$ of the solution $y(t)$:

$$Y(s) = \frac{(2s+1)y(0) + 2y'(0)}{2s^2 + s + 2} + \frac{G(s)}{2s^2 + s + 2} = \frac{G(s)}{2s^2 + s + 2}\,,$$

where $G(s) = \mathcal{L}\{g(t)\}$. To find $G(s)$, we observe that we can write

$$g(t) = u_5(t) - u_{20}(t)\,.$$

By properties of the Laplace transform that we established in the previous lecture, we have:

$$G(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = \frac{e^{-5s} - e^{-20s}}{s}\,, \quad s > 0\,.$$

Therefore,

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}\,.$$

Let $H(s) = 1/s(2s^2 + s + 2)$. Then

$$Y(s) = e^{-5s}H(s) - e^{-20s}H(s)\,,$$

so that if we can find $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we can apply properties of the inverse Laplace transform to get:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{e^{-5s}H(s)\right\} - \mathcal{L}^{-1}\left\{e^{-20s}H(s)\right\} = \boxed{u_5(t)h(t-5) - u_{20}(t)h(t-20)}\,. \tag{17.3}$$

In order to take the inverse Laplace transform of $H(s)$, we will need to use a partial fraction decomposition:

$$H(s) = \frac{1}{s(2s^2 + s + 2)} = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{s + 1/2}{2s^2 + s + 2} = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s + 1/2}{s^2 + (1/2)s + 1}\right)\,.$$

Completing the square:

$$s^2 + (1/2)s + 1 = \left(s + \frac{1}{4}\right)^2 + \frac{15}{16}\,.$$

Moreover, we can write

$$s + \frac{1}{2} = s + \frac{1}{4} + \frac{1}{4} = s + \frac{1}{4} + \frac{1}{\sqrt{15}}\left(\sqrt{\frac{15}{16}}\right)$$

Taking these results together, we get:

$$H(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s+1/4}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right) - \frac{1}{2\sqrt{15}}\left(\frac{\sqrt{15/16}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right),$$

which has inverse

$$\boxed{h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\sqrt{15}t/4\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\sqrt{15}t/4\right)}$$

Applying this result to what we found in (17.3), we have found the solution to the given IVP (Figure 7).

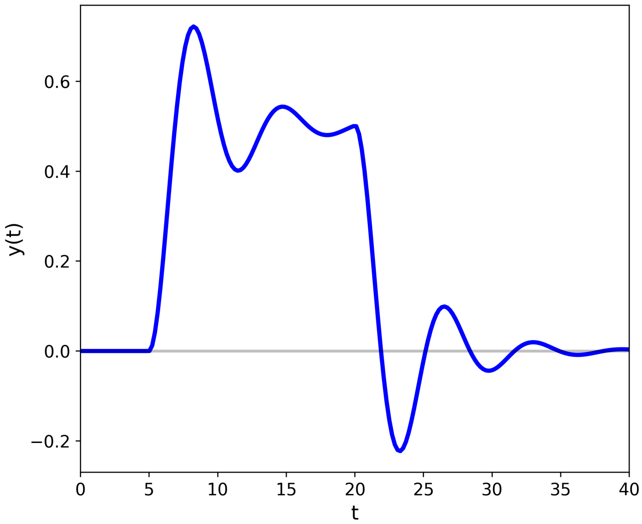


Figure 7: Plot of the solution $y(t)$ to (17.1).

Let's consider the plot of this solution, seen in Figure 7. Qualitatively, it consists of three distinct parts, corresponding to the intervals $(0,5)$, $(5,20)$, and $(20,\infty)$. By how the function $g(t)$ is defined, the ODE (17.1) on the interval $0 < t << 5$ is given by

$$2y'' + y' + 2y = 0\,,$$

with the given initial conditions $y(0) = 0$ and $y'(0) = 0$. Therefore, on this interval we would expect our solution $y(t)$ to be the solution to this homogeneous equation with the given initial conditions. As it turns out, the solution to this IVP is exactly $y = 0$, and we see that our solution agrees with this on the interval $0 < t < 5$.

What happens at $t = 5$? This is where we have our first jump in the forcing function $g(t)$. On the interval $5 < t < 20$, the ODE (17.1) reads

$$2y'' + y' + 2y = 1\,.$$

As for initial conditions, we evaluate the solution we found above in the limit $t \to 5^-$ to get that that

$$y(5) = 0 \quad y'(5) = 0\,.$$

The solution to this IVP is equal to the solution of the homogeneous equation (as it turns out, decaying, or *damped*, oscillations) plus a particular solution of this non-homogeneous equation (think back to our rules method of undermined coefficients: $y_P =$ constant, in this case equal to $1/2$). Looking at our plot of the solution we see exactly this behavior: oscillation about a constant value of $1/2$ with amplitude tending to zero.

Finally, at $t = 20$, the forcing function $g(t)$ jumps back down to zero. Therefore, on the interval $t > 20$, the ODE (17.1) is once again a homogeneous equation:

$$2y'' + y' + 2y = 0\,,$$

this time with initial conditions

$$y(20) \approx 0.50162, \quad y'(20) \approx 0.01125\,.$$

The solution to this IVP is a function that oscillates about $y = 0$ and has decaying amplitude; we see this behavior reflected in our solution for $t > 20$.

Remark. Separating our IVP into three separate IVPs over three separate intervals helped us visualize and better understand the qualitative behavior of our solution. However, as you can see, it would be quite tedious to solve every problem with a discontinuous $g(t)$ in this manner. This is why the Laplace transform is so useful in this case: we could find the exact solution to the IVP with less work!

How else do the discontinuities in $g(t)$ affect our solution $y(t)$? We can see that the function $h(t)$ and its derivatives are continuous functions. Therefore, the only possible discontinuities in $y(t)$ are at our "jump" points, $t = 5$ and $t = 20$. If we were to write $y(t)$ out in full, we could show that $y(t)$ and $y'(t)$ are continuous *even at these points.* However, it can also be shown that

$$\lim_{t\to 5^-} y''(t) = 0, \quad \text{and} \quad \lim_{t\to 5^+} y''(t) = \frac{1}{2}$$

which means that the second derivative of our solution is *not continuous* at $t = 5$; similarly, it can be shown that it is no continuous at $t = 20$ either. In other words, **the jump in the forcing term is "balanced" out** by a corresponding jump in the highest order term of the ODE, $2y''$.

Remark. Consider the second order linear equation

$$y'' + p(t)y' + q(t)y = g(t)\,,$$

where p and q are continuous on some interval $a < t < b$, but g is only *piecewise continuous* on that interval. Then a solution $y = \phi(t)$ to this equation is continuous with continuous first derivative, but ϕ'' will have jump discontinuities at the same points as g. This phenomenon is generalizable to higher order linear ODEs: the highest order derivative appearing in the equation will have jump discontinuities at the same points as the forcing function, but the lower order derivatives and the solution itself will be continuous.

Example 17.2. Now, let us consider the IVP:

$$y'' + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 0\,, \tag{17.4}$$

where

$$g(t) = \begin{cases} 0 & 0 \leqslant t < 5 \\ \dfrac{t-5}{5} & 5 \leqslant t < 10 \\ 1 & t \geqslant 10 \end{cases}$$

This function is a type of *ramp function.* These sorts of functions show up in applications in electrical engineering, as well as in the design of neural networks. In general a ramp function $R_c(t)$ is written as

$$R_c(t) = u_c(t)(t - c)\,.$$

Using properties of the Laplace transform,

$$\mathcal{L}\left\{R_c(t)\right\} = e^{-cs}\mathcal{L}\left\{t\right\} = \frac{e^{-cs}}{s^2}\,.$$

Then we can write

$$g(t) = \frac{u_5(t)(t-5) - u_{10}(t)(t-5)}{5} = \frac{R_5(t) - R_{10}(t)}{5}\,,$$

which has Laplace transform

$$G(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}\,.$$

Then the Laplace transform $Y(s)$ of the solution $y(t)$ to (17.4) is

$$Y(s) = \frac{G(s)}{s^2+4} = \left(\frac{e^{-5s} - e^{-10s}}{5}\right)\frac{1}{s^2(s^2+4)} =: \left(\frac{e^{-5s} - e^{-10s}}{5}\right)H(s)\,.$$

As we did in the last example, if we let $h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}$, then we have that

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{e^{-5s}H(s) - e^{-10s}H(s)\right\} = \frac{u_5(t)h(t-5) - u_{10}(t)h(t-10)}{5}$$

Thus, we need to find the inverse Laplace transform of $H(s) = 1/s^2(s^2+4)$; we do so using a partial fraction decomposition:

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4} = \frac{1/4}{s^2} - \frac{1}{8}\cdot\frac{2}{s^2+4} \implies \boxed{h(t) = \frac{t}{4} - \frac{1}{8}\sin(2t)}\,.$$

Looking at the plot of the solution $y(t)$, we see that $y = 0$ for $0 \leqslant t < 5$ (Figure 8). It then oscillates about a linear function for $5 < t < 10$, and oscillates about $y = 1/4$ with constant amplitude for $t > 10$ (Figure 8). If we were to go about separating the given IVP into three IVPs according to the piecewise definition of $g(t)$, as we did in the previous example, we would see that this behavior matches up with the three solutions to these three different IVPs.

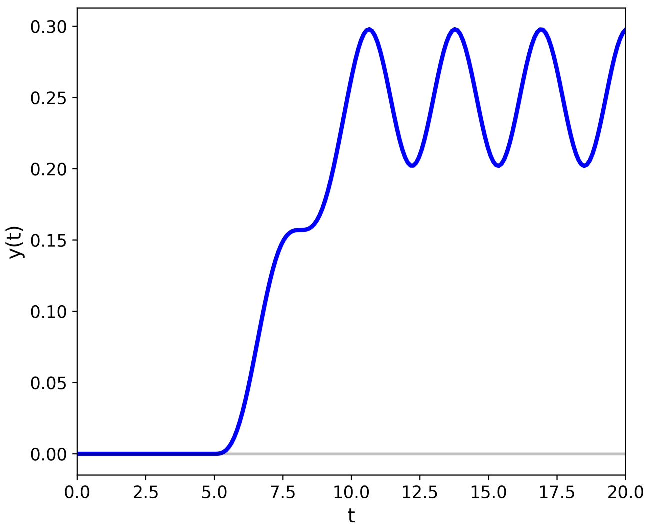


Figure 8: Plot of the solution $y(t)$ to (17.4).

Lecture 18: Impulse Functions

These lecture notes were adapted from Chapter 6.5 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

18.1 Introduction

In the last lecture, we saw how to solve IVPs of the form,

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = v_0\,,$$

where $g(t)$ was a discontinuous function that could be written in terms of the unit step function. In applications, $g(t)$ often represents some external voltage or physical force that is acting on a given system. Today, we are going to learn how to solve problems when the phenomenon modeled by $g(t)$ is *impulsive* in nature, e.g., an external force of large magnitude that acts on the system over a very short time interval.

18.2 Dirac delta function

In particular, let us consider the forcing function $g(t) = d_\tau(t)$ defined by:

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & -\tau < t < \tau \\ 0 & t \leqslant -\tau, \quad t \geqslant \tau \end{cases}\,,$$

where $\tau > 0$ (Figure 9). Define the integral $I(\tau)$ as

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\,dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt\,.$$

We note that $I(\tau) = 1$ so long as $\tau > 0$. This integral measures the overall *strength* of the forcing function $d_\tau(t)$; we call $I(\tau)$ the total **impulse** of the force $d_\tau(t)$ over the time interval $(-\tau, \tau)$.

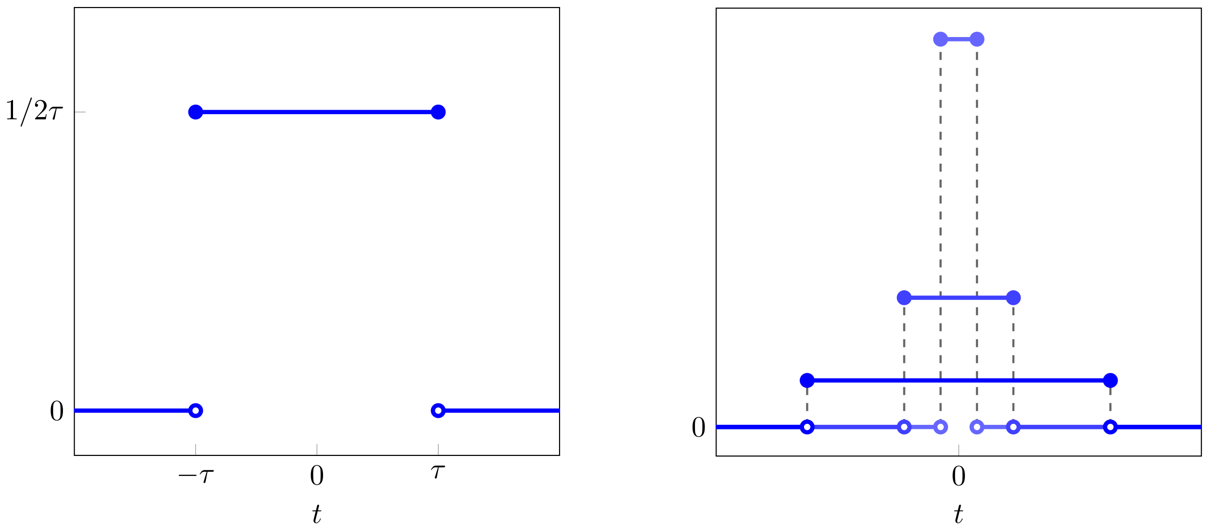


Figure 9: Graph of the forcing function $d_\tau(t)$ for fixed τ (left) and in the limit $\tau \to 0^+$ (right).

Suppose we take $\tau \to 0^+$, i.e., we consider a force d_τ with increasingly large magnitude $1/2\tau$ that acts over an increasingly small interval $-\tau < t < \tau$ (Figure 9). In fact, we get that:

$$\lim_{\tau \to 0^+} d_\tau(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}\,, \tag{18.1}$$

On the other hand, since $I(\tau) = 1$ for all $\tau > 0$,

$$\lim_{\tau \to 0^+} I(\tau) = 1\,. \tag{18.2}$$

Therefore, in the limit as $\tau \to 0^+$, d_τ is a force that is equal to zero except at one point (where it becomes arbitrarily large), but still has total impulse equal to one.

We can use these two conditions – namely, Eqs. (18.1) and (18.2) – to define an idealized **unit impulse function**, sometimes called the **Dirac delta function**. That is, the "function" $\delta(t)$ is defined through the following properties:

$$\delta(t) = 0 \qquad t \neq 0\,; \qquad\qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1\,. \tag{18.3}$$

There is no "normal" function (i.e., the kind you have seen in Calc I/II/III) that satisfies these properties. In fact, $\delta(t)$ defined through these equations is known as a *generalized function*. The Dirac delta function $\delta(t)$ corresponds to a unit impulse at $t = 0$, but we can easily extend this definition to a unit impulse at an arbitrary point $t = t_0$:

$$\delta(t - t_0) = 0 \qquad t \neq t_0\,; \qquad\qquad \int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1\,. \tag{18.4}$$

18.2.1 Laplace transform

The delta function does not satisfy the conditions of our Existence Theorem for the Laplace transform (namely, it is not piecewise continuous because of the "infinitely large" jump at $t = t_0$). However, its Laplace transform can be formally defined. Recall that we defined δ as the limit of the piecewise continuous function d_τ in the limit $\tau \to 0^+$. Then it makes sense to define the Laplace transform of δ in a similar manner.

In particular, we assume that $t_0 > 0$, and define $\mathcal{L}\{\delta(t - t_0)\}$ by

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \to 0^+} \mathcal{L}\{d_\tau(t - t_0)\}\,. \tag{18.5}$$

How do we compute this Laplace transform? Namely, how do we evaluate the limit in (18.5)? First, we recall the definition of $d_\tau(t - t_0)$:

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & t_0 - \tau < t < t_0 + \tau \\[2ex] 0 & t \leqslant t_0 - \tau, \quad t \geqslant t_0 + \tau \end{cases},$$

We also observe that because $t_0 > 0$, in the limit $\tau \to 0^+$, we will eventually have $0 < \tau < t_0$. That is, $t_0 - \tau > 0$. Thus, for these values of τ, we can write

$$d_\tau(t) = \frac{u_{t_0-\tau}(t) - u_{t_0+\tau}(t)}{2\tau}\,.$$

Then we have:

$$\begin{aligned}\mathcal{L}\{d_\tau(t - t_0)\} &= \frac{1}{2\tau}\Big(\mathcal{L}\{u_{t_0-\tau}(t)\} - \mathcal{L}\{u_{t_0+\tau}(t)\}\Big) = \frac{1}{2\tau}\left(\frac{e^{-(t_0-\tau)s}}{s} - \frac{e^{-(t_0+\tau)s}}{s}\right) \\ &= \frac{e^{\tau s} - e^{-\tau s}}{2\tau s} e^{-t_0 s} = \frac{\sinh(\tau s)}{\tau s} e^{-t_0 s}\end{aligned}$$

Using L'Hôpital's rule,

$$\lim_{\tau\to 0^+} \frac{\sinh(\tau s)}{\tau s} = \lim_{\tau\to 0^+} \frac{s\cosh(\tau s)}{s} = \cosh(0) = 1\,.$$

Therefore,

$$\mathcal{L}\left\{\delta(t-t_0)\right\} = \lim_{\tau\to 0^+} \mathcal{L}\left\{d_\tau(t-t_0)\right\} = e^{-t_0 s} \lim_{\tau\to 0^+} \frac{\sinh(\tau s)}{\tau s} = e^{-t_0 s}\,.$$

Recall that we began by assuming $t_0 > 0$. How do we find $\mathcal{L}\left\{\delta(t)\right\}$? Another limit!

$$\mathcal{L}\left\{\delta(t)\right\} = \lim_{t_0\to 0^+} \mathcal{L}\left\{\delta(t-t_0)\right\} = \lim_{t_0\to 0^+} e^{-t_0 s} = 1\,.$$

Taken together, we can say that for any $c \geqslant 0$,

$$\boxed{\mathcal{L}\left\{\delta(t-c)\right\} = e^{-cs}}\,.$$

Remark. In a similar manner, we can define the integral of $\delta(t-t_0)f(t)$ for any continuous f, not just $f(t) = e^{-st}$. We have:

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\,dt = \lim_{\tau\to 0^+} \int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)\,dt = \lim_{\tau\to 0^+} \left[\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt\right]\,.$$

The mean value theorem for integrals tells us that

$$\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt = \big(t_0+\tau-(t_0-\tau)\big)f(t^*) = 2\tau f(t^*)\,,$$

for some point t^* in the interval $(t_0-\tau, t_0+\tau)$. Therefore,

$$\int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)\,dt = f(t^*)\,.$$

In the limit $\tau \to 0^+$, the arbitrary point t^* between $t_0-\tau$ and $t_0+\tau$ must approach $t^* = t_0$ (by the "squeeze theorem"). Therefore,

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\,dt = \lim_{\tau\to 0^+} \int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)\,dt = f(t_0)\,.$$

This is an important property of the Dirac delta function.

Remark. So far, we have only been considering the unit step function $u_c(t)$ defined for $t \geqslant 0$, as this is the definition that is most useful to us in the study of Laplace transforms. We could just as easily define it on the entire real line:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geqslant c \end{cases}$$

where c can take on any value (i.e., is not restricted to $c \geqslant 0$). Let's consider the derivative of $u_c(t)$, $u_c'(t)$. It is certainly equal to zero for all $t \neq c$, as $u_c(t)$ is constant on those intervals. It also is "infinitely large" at the jump point $t = c$. Furthermore,

$$\int_{-\infty}^{\infty} u_c'(t)\,dt = \lim_{b\to\infty} u_c(b) - \lim_{a\to-\infty} u_c(a) = 1 - 0 = 1\,.$$

Taken together, we have that $u_c'(t)$ satisfies

$$u_c'(t) = 0, \quad t \neq c; \qquad \int_{-\infty}^{\infty} u_c'(t)\, dt = 1\,.$$

That is, $\boxed{u_c'(t) = \delta(t-c)}$.

18.3 IVPs with impulsive forcing functions

Example 18.1. Consider the IVP:

$$2y'' + y' + 2y = \delta(t-5), \quad y(0) = 0, y'(0) = 0\,. \tag{18.6}$$

We can use what we have just established about the delta function and its Laplace transform to solve this problem. Using our formula for $Y(s) = \mathcal{L}\{y(t)\}$, we have:

$$Y(s) = \frac{\mathcal{L}\{\delta(t-5)\}}{2s^2+s+2} = \frac{e^{-5s}}{2s^2+s+2}\,.$$

As we saw in the previous lecture, we can write:

$$2s^2+s+2 = 2\left[\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}\right]\,.$$

Then our expression for $Y(s)$ becomes:

$$Y(s) = \frac{e^{-5s}}{2}\left(\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right) =: \frac{e^{-5s}}{2}H(s)\,.$$

Using known inverse Laplace transforms, we know that:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{4}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)\,.$$

Then by properties of the Laplace transform, we have:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}u_5(t)h(t-5) = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\left[\frac{\sqrt{15}}{4}(t-5)\right]$$

Looking at this expression, we can already see that our solution will be equal to zero until the impulse at $t = 5$ because of the unit step function that multiplies the entire exprssion). After this point the solution oscillates with period $8\pi/\sqrt{15}$ and exponentially decreasing amplitude. A plot of this solution is given in Figure 10.

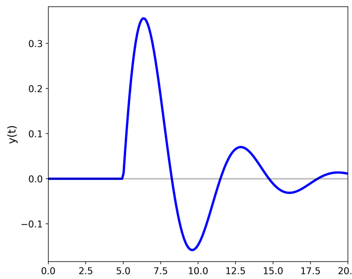


Figure 10: Solution $y(t)$ of the IVP (18.6).

Lecture 19: Intro to Systems of ODEs

These lecture notes were adapted from Chapter 7.1 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition).

19.1 Introduction

We will now begin our study of *systems of differential equations*. Such equations are relations between a number of dependent variables $x_1, x_2, \ldots, x_n$, their derivatives, and and independent variable t. Systems of ODEs often arise in applications, and are best illustrated with some examples.

Example 19.1 (Lotka-Volterra Model)**.**

This example illustrates a model of the relationship between the population size of a predator species and the population size of a prey species called the **Lotka-Volterra equations**.

Gray wolves have historically been found throughout Washington state, but because they were hunted to near extinction in the 1930s, they are only found in a handful of areas in the state today. Understanding how these populations interact with populations of prey species (elk, deer) as well as other predator species (bobcats, cougars, coyotes) is an important problem when it comes to managing public lands in a way that conserves these populations.[2]

Let's consider a simple model of the interaction between a gray wolf population and an elk population in a given area. Let x_1 denote the *population density* of the wolves (i.e., number of wolves per square mile), and let x_2 denote the population density of the elk. Under some assumptions on the behavior of each of the populations, we can model these populations via the following system of ODEs:

$$\begin{cases} x_1' = -ax_1 + bx_1x_2 \\ x_2' = cx_2 - dx_1x_2 \end{cases}$$

The coefficients a, b, c, and d are all positive. The terms $-ax_1$ and cx_2 represent the change in the wolf and elk populations, respectively, due to birth, death, and/or migration. The terms bx_1x_2 and dx_1x_2 represent the change in the wolf and elk populations, respectively, due to interactions between the two species (i.e., predation). We note that this system is **first order, autonomous**, and **nonlinear**.

Example 19.2 (Spring-mass System)**.**

Consider the *spring-mass system* given in Figure [x], which consists of two masses with mass m_1 and m_2, respectively, connected by three springs that have *spring constants* k_1, k_2, and k_3, respectively. The two masses in this system move on a frictionless surface subject to some external forces $F_1(t)$ and $F_2(t)$; their motion is constrained by the three springs. Let x_1 and x_2 denote the *position* (i.e., horizontal displacement from resting position) of each of the two masses in this system. Then by Newton's second law, x_1 and x_2 satisfy the following second order ODEs:

$$\begin{cases} m_1x_1'' = k_2(x_2 - x_1) - k_1x_1 + F_1(t) \\ m_2x_2'' = -k_3x_2 - k_2(x_2 - x_1) + F_2(t) \end{cases}$$

The term $k_2(x_1 - x_2) - k_1x_1$ represents the *restoring force* of the second and first springs acting on the first mass. The term $-k_3x_2 - k_2(x_2 - x_1)$ represents the restoring force of the third and second springs acting on the second mass. We note that this system is **second order** and **linear**.

[2] The Washington Predator-Prey Project was a 5 year study started in 2016 by the Washington Department of Fish and Wildlife (WDFW) and the University of Washington to understand the effect of gray wolves and other predator species on deer and elk populations on WDFW-managed lands.

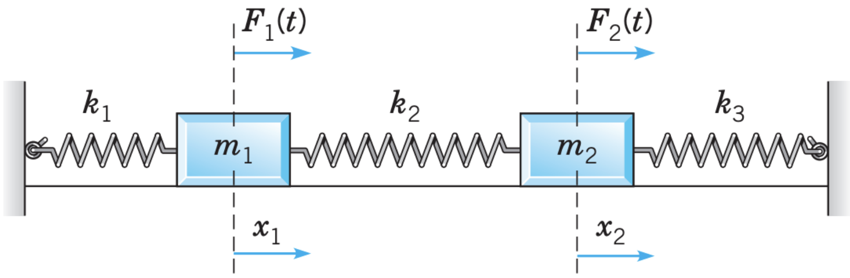


Figure 11: Diagram of a two-mass, three spring system. Figure from *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima.

19.2 First order systems

In this course, we will only consider first order systems of ODEs, i.e., systems in which the highest order derivative that appears in any of the equations is the first derivative. In general, a first order system of n ODEs is written as:

$$\begin{cases} x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_n) \end{cases} \tag{19.1}$$

Classifying systems of ODEs is similar to classifying ODEs. For example, we say that the above system is **linear** if every function $f_1, f_2, \ldots, f_n$ in the above system is linear in the dependent variables $x_1, x_2, \ldots, x_n$. We say that it is **autonomous** if the functions $f_1, f_2, \ldots, f_n$ only depend on t through the dependent variables $x_1, x_2, \ldots, x_n$.

Oftentimes, a system of the form (19.1) is written in *vector form*. Define the n-vector $\mathbf{x}$ as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{where} \quad \mathbf{x}' := \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}.$$

Define the vector field $\mathbf{f}$ as:

$$\mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_2(t, \mathbf{x}) \end{pmatrix}.$$

Then we can write the system (19.1) as

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$$

When the independent variable t represents *time*, we call this system a *dynamical system*.

19.2.1 Systems and higher order ODEs

One reason why we are only studying first order systems is that any equation of higher order can always be transformed into a system of first order ODEs. To see this, consider any ODE of order n:

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}). \tag{19.2}$$

Define the following dependent variables:

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \ldots, \quad x_n = y^{(n-1)}.$$

Then we have

$$x_1' = y' = x_2, \quad x_2' = y'' = x_2, \quad \dots, \quad x_n' = y^{(n)} = f(t, x_1, x_2, \dots, x_n).$$

Thus, (19.2) can be written as the following system of n first order ODEs:

$$\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_n' &= f(t, x_1, x_2, \dots, x_n) \end{cases}$$

Example 19.3. Consider the general equation of harmonic oscillator, which was derived on HW4:

$$my'' + \gamma y' + ky = F(t). \tag{19.3}$$

Let $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$, and $x_2' = y''$, where

$$y'' = \frac{F(t) - \gamma y' - ky}{m} = \frac{F(t) - \gamma x_2 - kx_1}{m}.$$

Then (19.3) can be written as the following first order system:

$$\begin{cases} x_1' = x_2 \\ x_2' = \dfrac{F(t) - \gamma x_2 - kx_1}{m} \end{cases} \tag{19.4}$$

What if we were given initial conditions $y(t_0) = y_0$ and $y'(t_0) = v_0$? Then the corresponding initial conditions for the system (19.4) would be $x_1(t_0) = y_0$ and $x_2(t_0) = v_0$. We can write this IVP in vector form as

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x_0},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{x_0} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}, \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ \dfrac{F(t) - \gamma x_2 - kx_1}{m} \end{pmatrix}.$$

Lectures 20-21: First Order Autonomous Systems

These lecture notes were adapted from Chapter 9.2 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition) and the course notes of Dr. Ami Radunskaya (Pomona College).

20.1 Introduction

Recall the general form of a *first order* system of ODEs in n dependent variables $x_1(t), x_2(t), \ldots, x_n(t)$ of the independent variable t:

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' = f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \qquad\qquad \vdots \\ x_n' = f_n(t, x_1, x_2, \ldots, x_n) \end{cases} . \tag{20.1}$$

We can simplify how we write this system by introducing some *vector notation*:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} .$$

We say that $\mathbf{x}(t)$ is a *vector-valued function*, and that $x_i(t)$ is the ith component of $\mathbf{x}(t)$ for $i = 1, 2, \ldots, n$. In terms of the derivative of the function $\mathbf{x}(t)$, we write:

$$\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} .$$

Define the vector-valued function $\mathbf{f}(t, \mathbf{x})$ as

$$\mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix} ,$$

where we have introduced the notation $f_i(t, \mathbf{x}) = f_i(t, x_1, x_2, \ldots, x_n)$. Then we can write the first order system (20.1) as a first order ODE of vector-valued functions:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) . \tag{20.2}$$

Remark. In writing the equation (20.2), we relied on the definition of what it means for two vectors to be equal: given two vectors $\mathbf{v} = (v_1 \ v_2 \ \ldots \ v_n)$ and $\mathbf{w} = (w_1 \ w_2 \ \ldots \ w_n)$ of the same dimension n, we say that $\mathbf{v} = \mathbf{w}$ if $v_i = w_i$ for each index $i = 1, 2, \ldots, n$.

Some terminology for systems of ODEs:

- We call the individual dependent variables $x_1, x_2, \ldots, x_n$ the **state variables** of our system. The vector $\mathbf{x} = (x_1 \ x_2 \ \ldots \ x_n)$ is the **state vector** of the system.
- The space in which our state vector "lives" (i.e., is an element of) is call the **state space** (or sometimes **phase space**) of the system. If $n = 1$, state space is the real line, denoted $\mathbb{R}$ (a.k.a. *phase line*); if $n = 2$, state space is the real plane, denoted $\mathbb{R}^2$ (a.k.a. the *phase plane*).
- In state space, the **solution curves** $\mathbf{x}(t)$ corresponding to different initial conditions $\mathbf{x}(0)$ are called **orbits** or **trajectories** of the system. The graphs of the individually plotted solution curves $\big(t, x_i(t)\big)$ are called the **component curves** of the system.

20.1.1 Classifying first order systems

Compare the vector equation (20.2) to the general form of a first order ODE for unknown $x(t)$:

$$x' = f(t, x)\,.$$

These look similar! As with first order ODEs, we can classify first order systems depending on the properties of the function $\mathbf{f}(t, \mathbf{x})$ in (20.2).

- We say that the system (20.2) **linear** if $\mathbf{f}(t, \mathbf{x})$ is linear in $\mathbf{x}$, i.e., if we can write $\mathbf{f}(t, \mathbf{x}) = P(t)\mathbf{x} + \mathbf{g}(t)$ for some $n \times n$ matrix $P(t)$ whose entries vary with t and some vector valued function $\mathbf{g}(t)$. The multiplication of an $n \times n$ matrix and an n vector is defined as the n vector whose ith component is equal to the dot product of the vector and row i of the matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}$$

  We note that this definition of a linear system is equivalent to being able to write each component $f_i(t, \mathbf{x})$ of $\mathbf{f}(t, \mathbf{x})$ as

$$f_i(t, \mathbf{x}) = a_{i0}(t) + \sum_{k=1}^{n} a_{ik}(t)x_k$$

  for some coefficient functions $a_{i0}(t), a_{i1}(t), \ldots, a_{in}(t)$. From this definition, we can define

$$\mathbf{g}(t) = \begin{pmatrix} a_{10}(t) \\ a_{20}(t) \\ \vdots \\ a_{n0}(t) \end{pmatrix} \quad \text{and} \quad P(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

  and arrive at the other definition.

- A linear system is **homogeneous** if $\mathbf{g}(t) = \mathbf{0}$, where $\mathbf{0}$ is the vector of all zeros.
- We say that the system (20.2) **autonomous** if $\mathbf{f}(t, \mathbf{x}) = \mathbf{f}(\mathbf{x})$. That is, a system is autonomous if $\mathbf{x}'(t)$ only depends on t through $\mathbf{x}(t)$.

Note: By these definitions, systems that are linear, homogeneous, and autonomous are exactly those that can be written as

$$\mathbf{x}' = A\mathbf{x}$$

for some $n \times n$ matrix A with constant entries. These systems are a special class of systems.

20.2 Autonomous Systems

In the following lectures, we will be focusing on first order systems that are autonomous:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})\,. \tag{20.3}$$

In particular, we will be focusing on *two-dimensional* (2D) systems ($n = 2$):

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}.$$

Similar to autonomous ODEs, we can analyze the *qualitative* behavior of solutions to autonomous first order systems without having to solve them.

Recall that in 1D, we could analyze the behavior of solutions of any ODE using the direction field (t, x'). In the special case of autonomous ODEs, we could simplify this analysis onto the *phase line* by finding the fixed points of the ODE and classifying their stability based on the sign of $x' = f(x)$ nearby. We can do a similar analysis in the *phase plane* for 2D autonomous systems. For these types of systems, a plot of sample trajectories in the phase plane along with a sketch of the direction field is called a **phase portrait** of the system.

> *Definition.* A point $\mathbf{x} = \mathbf{x}^*$ is a **fixed point** of the autonomous system (20.3) if
>
> $$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}\,.$$

Remark. As with the 1D case, these points are important because they correspond to $\mathbf{x}' = \mathbf{0}$. That is, they are *constant solutions* to the autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. Because $\mathbf{x}'$ on depends on the position $\mathbf{x}$ of the system, solutions for initial conditions $\mathbf{x}(t_0) = \mathbf{x}^*$ remain at $\mathbf{x}^*$ for all $t > t_0$.

Remark. Another way to express the fixed point condition is $\mathbf{x}^*$ such that

$$\begin{cases} f_1(\mathbf{x}^*) = 0 \\ f_2(\mathbf{x}^*) = 0 \\ \quad\vdots \\ f_n(\mathbf{x}^*) = 0 \end{cases}$$

What about the points $\mathbf{x}^*$ that only satisfy $f_i(\mathbf{x}^*) = 0$ for some fixed $i \in \{1, 2, \ldots, n\}$?

> *Definition.* The x_i-**nullcline** of (20.3) is the collection of all points $\mathbf{x}$ that satisfy
>
> $$f_i(\mathbf{x}) = 0\,.$$

Note: The fixed points of a system lie at the intersection of the nullclines of the system. In 2D, these nullclines are curves in the phase plane.

Before we discuss how to classify the fixed points by their *stability*, let us demonstrate these new concepts with an example.

Example 20.1. Consider the 2D autonomous system

$$\begin{cases} x_1' = -(x_1 - x_2)(1 - x_1 - x_2) \\ x_2' = x_1(2 + x_2) \end{cases}.$$

The fixed points of this system are the points (x_1^*, x_2^*) that solve the system of algebraic equations

$$\begin{cases} 0 = -(x_1^* - x_2^*)(1 - x_1^* - x_2^*) \\ 0 = x_1^*(2 + x_2^*) \end{cases}.$$

Recall that these fixed points lie at the intersection of the x_1- and x_2-nullclines, i.e., the curves in the x_1x_2-plane along which $x_1' = 0$ and $x_2' = 0$, respectively.

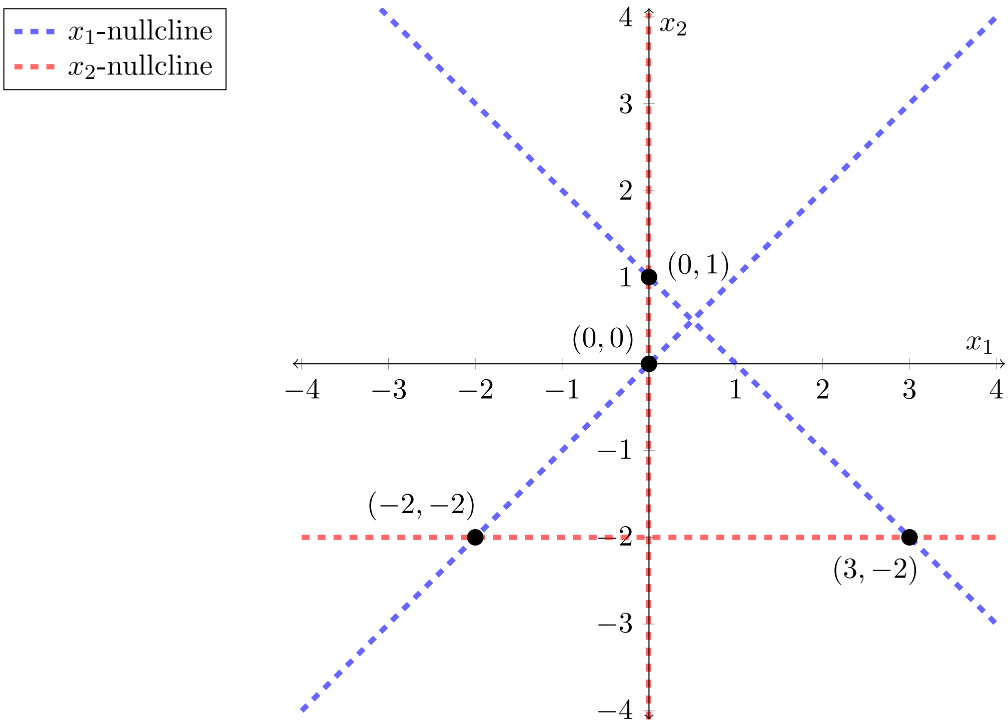
- The x_1-nullclines are the lines

$$\underset{(x_1 - x_2 = 0)}{x_1 = x_2} \qquad \text{and} \qquad \underset{(1 - x_1 - x_2 = 0)}{x_2 = -x_1 + 1}\,.$$

- The x_2-nullclines are the lines

$$x_1 = 0 \qquad \text{and} \qquad x_2 = -2\,.$$

Let's plot these nullclines in the phase plane:



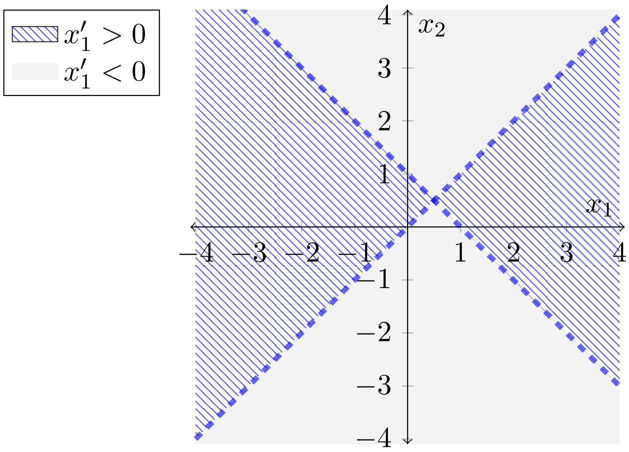
The intersections of the x_1-nullclines with the x_2-nullclines are the fixed points: $(0,0), (0,1), (-2,-2)$, and $(3,-2)$. How do we determine the stability of the fixed points? In 1D, our "system" was constrained to the phase *line*, so we could simply look at the sign of $x' = f(x)$ in between the fixed points to determine their stability. In 2D, we don't have the same concept of being "between" fixed points, as our system now "lives" in the 2D plane.

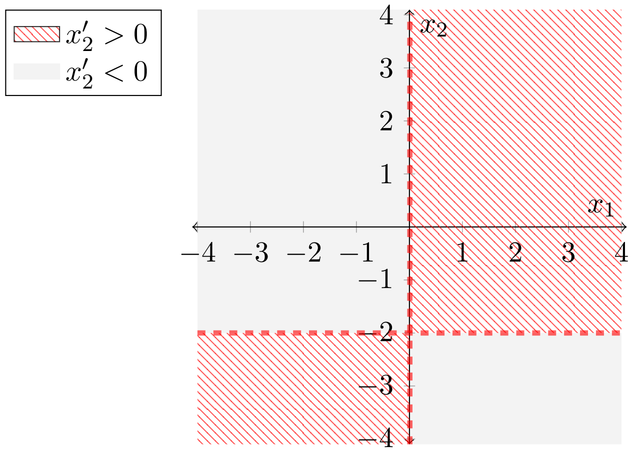
Instead, we will have look at the **direction field** of our 2D system:

$$\mathbf{f}(\mathbf{x}) = \big(f_1(x_1, x_2), f_2(x_1, x_2)\big).$$

As $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, this vector field gives the slope of the tangent line to any solution curve $\mathbf{x}(t)$ at the point $\mathbf{x}(t) = \mathbf{x}$. The direction field tells us how solutions change instantaneously at a given point, which means that we can use it as a guide to sketch some solutions of the given system (solutions "follow the arrows"). Thus, we can use the direction field to see how solutions behave near fixed points.

Returning to our example, let's add a sketch the direction field of our system to our phase portrait. We could do so by picking a handful of points at which to compute $\mathbf{f}(\mathbf{x})$ and draw in the corresponding arrows. However, it is useful to note that the two x_1-nullclines divide the plane into 4 regions based on the sign of x_1' (likewise with the x_2-nullclines and x_2'):





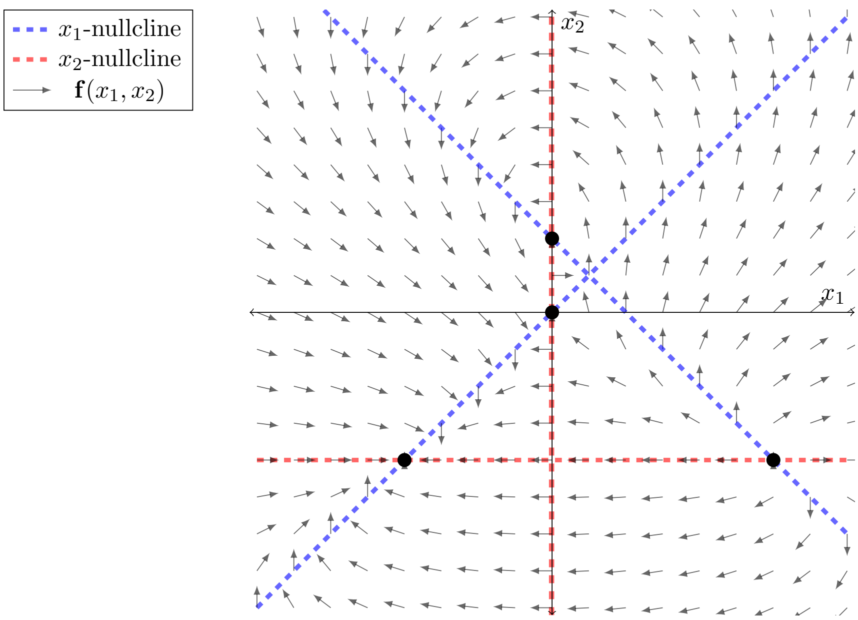
For $x_1' = -(x_1 - x_2)(1 - x_1 - x_2)$, we have that:

- $x_1' > 0$ when a) $x_2 > x_1$ and $x_2 < -x_1 + 1$ ("left" region above left) OR b) $x_2 < x_1$ and $x_2 > -x_1 + 1$ ("right" region above left)
- $x_1' < 0$ when a) $x_2 > x_1$ and $x_2 > -x_1 + 1$ ("upper" region above left) OR b) $x_2 < x_1$ and $x_2 < -x_1 + 1$ ("lower" region above left)

For $x_2' = x_1(2 + x_2)$, we have that:

- $x_2' > 0$ when a) $x_1 > 0$ and $x_2 > -2$ ("upper right" region above right) OR b) $x_1 < 0$ and $x_2 < -2$ ("lower left" region above right)
- $x_2' < 0$ when a) $x_1 < 0$ and $x_2 > -2$ ("upper left" region above right) OR b) $x_1 > 0$ and $x_2 < -2$ ("lower right" region above right)

We then use this information about the sign of x_1' and x_2' in each area of the phase plane to sketch the direction field (x_1', x_2') as arrows pointing in the direction based on the signs of x_1' and x_2'. That is to say, the more useful information to us when sketching a phase portrait is not the exact magnitude of the direction field but its *direction*. For this example, we have:



What does this direction field tell us?

- Along the x_1-nullclines, the direction field points upwards or downwards (as $x_1' = 0$). Similarly, along the x_2-nullclines, the direction field is horizontal (as $x_2' = 0$).
- *Sketching in solutions* that start near $\mathbf{x}^* = (-2, -2)$, we see that these solutions tend to stay near this point. In fact, they tend toward the fixed point. This behavior is similar to the behavior we saw in 1D with *asymptotically stable* fixed points.
- On the other hand, if we *sketch in solutions* that start near $\mathbf{x}^* = (3, -2)$, we see that these solutions tend away from the fixed point. This behavior is similar to the behavior we saw in 1D with *unstable* fixed points.
- If we *sketch in solutions* that start near $\mathbf{x}^* = (0, 0)$, we see that some solutions initially head towards this fixed point, but all of these solutions seem to tending away from it eventually. This behavior is similar to the behavior we saw in 1D with *semi-stable* fixed points.
- *Sketching in solutions* that start near $\mathbf{x}^* = (0, 1)$, we see that they "spiral" in counter-clockwise towards the fixed point. This behavior is unlike anything we have seen in 1D.

How do we classify the fixed points based on the observed behavior of "nearby" solutions? As alluded to above, we use the same general "definition" of stability as we did in 1D:

- A fixed point is **stable** if *all* solutions that start "near" it stay "near" for all t.
- If we additionally have that these solutions tend toward the fixed point as t increases, we say that the stable fixed point is **asymptotically stable**.
- A fixed point is **unstable** if it is not stable.

Using these loose definitions, we can immediately say that $\mathbf{x}^* = (-2, -2)$ and $\mathbf{x}^* = (0, 1)$ are asymptotically stable and $\mathbf{x}^* = (3, -2)$ and $\mathbf{x}^* = (0, 0)$ are unstable. The points $\mathbf{x}^* = (0, 1)$ and $\mathbf{x}^* = (0, 0)$ have special names: they are called a **stable spiral point** and a **saddle point**, respectively.

In the following lectures, we will establish a more rigorous theory about how to classify fixed points of autonomous 2D systems, starting with the linear homogeneous case. We note that, in general, we do not have to do this process of drawing the phase plane by hand. There are several software that we could use (`Python`, `MATLAB`, `pplane.jar`) to plot the nullclines, direction field, and several trajectories. The final lecture of this class will include a demo of exactly how to do this in Python.

Lectures 22-23: Linear Homogeneous Autonomous Systems

These lecture notes were adapted from Chapters 7 and 9.1 of the textbook *Elementary Differential Equations and Boundary Value Problems* by Boyce & DiPrima (9th edition) and the course notes of Dr. Ami Radunskaya (Pomona College).

22.1 Introduction

We will now consider a special case of autonomous systems of ODEs:

$$\mathbf{x}' = A\mathbf{x}\,, \tag{22.1}$$

where A is an $n \times n$ matrix with real, constant entries. We denote the space of all such matrices as $\mathbb{R}^{n\times n}$. The system (22.1) is linear, homogeneous, and autonomous, with $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$.

As we did before, we will focus here on the 2D case for simplicity. Many of the results established here generalize to higher dimensions. In two dimensions, the system $\mathbf{x}' = A\mathbf{x}$ becomes:

$$\underbrace{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}}_{\mathbf{x}'} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

Another way to write such a 2D system is

$$\begin{cases} x_1' = ax_1 + bx_2 \\ x_2' = cx_1 + dx_2 \end{cases}$$

Example 22.1. Consider the system

$$\begin{cases} x_1' = 3x_1 - 7x_2 \\ x_2' = 2x_2 \end{cases}$$

Identifying $a = 3, b = -7, c = 0$, and $d = 2$, we can write this system in matrix-vector form $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = (x_1\ x_2)$, and

$$A = \begin{pmatrix} 3 & -7 \\ 0 & 2 \end{pmatrix}.$$

22.2 Finding the fixed points of $\mathbf{x}' = A\mathbf{x}$

What are the fixed points of this system? By definition, they are the points $\mathbf{x}^*$ such that $A\mathbf{x}^* = \mathbf{0}$. From a linear algebraic standpoint, the set of all such vectors is called the **nullspace** of A. Linear algebra is not a prerequisite for this class, however, so for now we will proceed as we did in the previous lecture: finding the intersection of the nullclines.

For the system $\mathbf{x}' = A\mathbf{x}$, the x_1-nullcline is the line

$$ax_1 + bx_2 = 0\,, \tag{22.2}$$

and the x_2-nullcline is the line

$$cx_1 + dx_2 = 0\,. \tag{22.3}$$

Recall that two lines can either have

1. exactly one intersection,
2. infinitely many intersections (same line),
3. no intersection (parallel lines).

Thus, our system can have one fixed point, a line of fixed points, or no fixed points. (We note that if $a = b = c = d = 0$, then every point in the plane is a fixed point, but this would not be a very interesting system.) Both of the nullclines in (22.2) and (22.3) are lines that pass through the origin. Thus, the system $\mathbf{x}' = A\mathbf{x}$ has at least one fixed point: $\mathbf{x}^* = \mathbf{0} = (0, 0)$.

When do we have infinitely many fixed points? Ignoring, for now, the possibility that $b = 0$ or $d = 0$, we can write the two nullclines as

$$x_2 = -\frac{a}{b}x_1 \qquad\qquad x_2 = -\frac{c}{d}x_2$$

(x_1-nullcline) (x_2-nullcline)

These two lines are the same line when

$$\frac{a}{b} = \frac{c}{d} \iff \boxed{ad - bc = 0}.$$

This condition is a more general condition for infinitely many fixed points in the sense that it holds even if $b = 0$ or $d = 0$. The quantity $ad - bc = 0$ is called the **determinant** of A, denoted

$$\det(A) \qquad \text{or} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Remark. It can be shown that an equivalent condition to $ad - bc = 0$ is that $a = \alpha c$ and $b = \alpha d$ for some constant α. In vector form, this condition becomes

$$\begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} c \\ d \end{pmatrix}.$$

This condition looks a lot like our earlier definition of **linear dependence**, which we saw defined for two functions. That's because it is! In particular, the above statement is equivalent to saying that the *rows* of A are linearly dependent.

In sum, for the 2D system $\mathbf{x}' = A\mathbf{x}$:

- If $\det(A) = ad - bc \neq 0$, the system has exactly one fixed point at $\mathbf{x}^* = \mathbf{0}$.
- If $\det(A) = ad - bc = 0$, the system has infinitely many fixed points along the line $ax_1 + bx_2 = 0$.

Example 22.2. Consider the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix}.$$

We see that $\det(A) = 8 - 8 = 0$, which tells us that this system has infinitely many fixed points along the line $x_2 = -2x_1$.

Example 22.3. Consider the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 3 & 7 \\ -2 & 5 \end{pmatrix}.$$

We see that $\det(A) = 15 - (-14) = 29 \neq 0$, which tells us that this system has exactly one fixed point at $\mathbf{x}^* = \mathbf{0}$.

22.3 General solution of $\mathbf{x}' = A\mathbf{x}$

In the case that $\det(A) \neq 0$, how do we classify the fixed point $\mathbf{x}^* = \mathbf{0}$? Before, in the non-linear case, we relied on building a phase portrait of our system and looking at the behavior of solutions near the fixed point. Here, we can do a bit better: we can actually solve the system $\mathbf{x}' = A\mathbf{x}$.

To do so, first consider the 1D case:

$$x' = ax\,,$$

where $a \neq 0$. This ODE has solution $x(t) = Ce^{at}$, where C is an arbitrary constant. For the 2D case, let's try a similar approach: guess something of the form $e^{\lambda t}$, where λ is to be determined. Our solution must be a 2-vector, so instead of multiplying our guess by a constant C, we multiply it by a constant *vector* $\mathbf{v} = (v_1\ v_2)$. That is, our *candidate solution* is

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} = \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{pmatrix}\,. \tag{22.4}$$

We then plug this guess into $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v} \overset{set}{=} A\mathbf{x} = A(e^{\lambda t}\mathbf{v}) = e^{\lambda t}A\mathbf{v}\,.$$

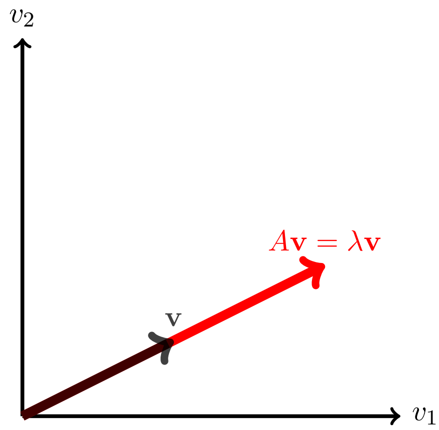
Dividing both sides by $e^{\lambda t} \neq 0$, we get a condition on λ and $\mathbf{v}$ such that $\mathbf{x}(t)$ in (22.4) is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$\boxed{\lambda\mathbf{v} = A\mathbf{v}}\,.$$

This condition is an important concept in linear algebra: any $\mathbf{v}$ that satisfies $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \neq 0$ is called an **eigenvector** of A with corresponding **eigenvalue** λ.

Note: This result holds for n-dimensional systems of the form $\mathbf{x}' = A\mathbf{x}$, not just 2D ones.

Remark. Geometrically, the eigenvectors of A are the vectors whose magnitudes are "stretched" or "shrunk" by A, but do not change direction.



Returning to our problem of solving $\mathbf{x}' = A\mathbf{x}$, we have shown that $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where $A\mathbf{v} = \lambda\mathbf{v}$, is a solution to this system of ODEs. How do we find $\mathbf{v}$ and λ? As it turns out, the eigenvalues of A are exactly the solutions of something called the **characteristic equation** of A:

$$(a - \lambda)(d - \lambda) - bc = 0\,. \tag{22.5}$$

The left-hand side of this equation is called the **characteristic polynomial** of A.

Remark. In general, for an $n \times n$ matrix A, the characteristic equation is $\det(A - \lambda I) = 0$, where I is the $n \times n$ identity matrix (the matrix with 1's on the diagonal and 0's elsewhere).

We can expand the polynomial in (22.5) as follows:

$$\begin{aligned} ad - a\lambda - d\lambda + \lambda^2 - bc = 0 \\ \lambda^2 - (a+d)\lambda + ad - bc = 0\,. \end{aligned}$$

We observe that $ad - bc = \det(A)$, and $a + d$ is the sum of the diagonal elements of A. We call the latter quantity the **trace** of A, denoted $\mathrm{tr}(A)$. Then, by the quadratic equation, the eigenvalues of a 2×2 matrix are:

$$\boxed{\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}}\,, \tag{22.6}$$

where $\det(A) = ad - bc$ and $\mathrm{tr}(A) = a + d$. Once we have the eigenvalues, we need only solve the linear algebraic equation $A\mathbf{v} = \lambda\mathbf{v}$ with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ to get the corresponding eigenvectors.

Example 22.4. Consider the following matrix A:

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}.$$

What are the eigenvalues and eigenvectors of this matrix? Using the formula in (22.6), the eigenvalues are:

$$\lambda = \frac{-1 \pm \sqrt{(-1)^2 - 4\big(1/4 + 1\big)}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = -\frac{1}{2} \pm i$$

Thus, this matrix has two eigenvalues that are complex. To find the corresponding eigenvectors, we solve $A\mathbf{v} = \lambda\mathbf{v}$ for $\mathbf{v}$ with $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

<u>$\lambda_1 = -\frac{1}{2} + i$.</u> The system of linear equations $A\mathbf{v} = \lambda_1\mathbf{v}$ reads:

$$A\mathbf{v} = \begin{pmatrix} -\frac{1}{2}v_1 + v_2 \\ -v_1 - \frac{1}{2}v_2 \end{pmatrix} \overset{set}{=} \begin{pmatrix} (-\frac{1}{2} + i)v_1 \\ (-\frac{1}{2} + i)v_2 \end{pmatrix} = \lambda_1\mathbf{v}\,.$$

Combining like terms on both sides of these equations, we get:

$$\begin{cases} -iv_1 + v_2 = 0 \\ -v_1 - iv_2 = 0 \end{cases}$$

We note that these are the same equation, up to multiplication/division on both sides by i:

$$i(-v_1 - iv_2) = i \cdot 0 \quad \xrightarrow{\text{becomes}} \quad -iv_1 + v_2 = 0\,.$$

Therefore, there are infinitely many solutions to this system of linear equations along the line

$$-iv_1 + v_2 = 0 \iff v_2 = iv_1\,.$$

In choosing a specific eigenvector, we can choose any vector that lies along this line. For simplicity, let us take $v_1 = 1$, so that $v_2 = i$. Then the eigenvector corresponding to eigenvalue $\lambda_1 = -\frac{1}{2} + i$ is

$$\boxed{\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}}.$$

<u>$\lambda_2 = -\frac{1}{2} - i$.</u> The system of linear equations $A\mathbf{v} = \lambda_2\mathbf{v}$ reads:

$$A\mathbf{v} = \begin{pmatrix} -\frac{1}{2}v_1 + v_2 \\ -v_1 - \frac{1}{2}v_2 \end{pmatrix} \overset{set}{=} \begin{pmatrix} (-\frac{1}{2} - i)v_1 \\ (-\frac{1}{2} - i)v_2 \end{pmatrix} = \lambda_2\mathbf{v}\,.$$

Combining like terms on both sides of these equations, we get:

$$\begin{cases} iv_1 + v_2 = 0 \\ -v_1 + iv_2 = 0 \end{cases}$$

We note that these are the same equation, up to multiplication/division on both sides by $-i$:

$$-i(-v_1 + iv_2) = i \cdot 0 \quad \xrightarrow{\text{becomes}} \quad iv_1 + v_2 = 0\,.$$

Therefore, there are infinitely many solutions to this system of linear equations along the line

$$iv_1 + v_2 = 0 \iff v_2 = -iv_1\,.$$

In choosing a specific eigenvector, we can choose any vector that lies along this line. For simplicity, let us take $v_1 = 1$, so that $v_2 = -i$. Then the eigenvector corresponding to eigenvalue $\lambda_1 = -\frac{1}{2} - i$ is

$$\boxed{\mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}}\,.$$

Remark. We can write $\mathbf{v}^{(1,2)}$ in terms of their real and complex parts:

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We note that the complex part of $\mathbf{v}^{(2)}$ is -1 times the complex part of $\mathbf{v}^{(1)}$; we say that $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are **complex conjugates**. Similarly, our two eigenvalues $\lambda_{1,2} = -\frac{1}{2} \pm i$ are complex conjugates. In general, when A has complex eigenvalues they are always complex conjugates, and their corresponding eigenvectors are also complex conjugates. That is, if A has complex eigenvalues, then they will be $\lambda_{1,2} = \eta \pm i\mu$ for some real numbers η and μ, and the eigenvectors will be $\mathbf{v}^{(1,2)} = \mathbf{a} \pm i\mathbf{b}$ for some real vectors $\mathbf{a}$ and $\mathbf{b}$.

Suppose we wanted to solve the system $\mathbf{x}' = A\mathbf{x}$, where A is the matrix we just saw in Example 22.4. From this example, we know that this matrix has two distinct, complex eigenvalues $\lambda_{1,2}$ with corresponding eigenvectors $\mathbf{v}^{(1,2)}$. Then we have found two solutions to $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\mathbf{v}^{(1)} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\mathbf{v}^{(2)}\,.$$

How do we get from these two particular solutions to the general solution to $\mathbf{x}' = A\mathbf{x}$? Think back to the last time in this course we saw two "distinct" solutions to an ODE that we derived from a characteristic equation: 2nd order, linear, homogeneous ODEs. As it turns out, we take a very similar approach here as we did with these 2nd order ODEs.

Remark. This parallel shouldn't surprise us, as we know that we can write a linear 2nd order ODE as a 2D linear system of first order equations. In particular, given the ODE

$$ay'' + by' + cy = 0\,, \tag{22.7}$$

the corresponding 2D system with $x_1 = y$ and $x_2 = y'$ is

$$\mathbf{x}' = A\mathbf{x}\,, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}. \tag{22.8}$$

The characteristic equation of this matrix A is

$$\lambda^2 - \left(-\frac{b}{a}\right)\lambda + \frac{c}{a} = 0 \quad \xrightarrow{\text{multiply by } a} \quad a\lambda^2 + b\lambda + c = 0\,,$$

which is exactly the characteristic equation of the 2nd order ODE in (22.7). Therefore, the eigenvalues of A in (22.8) are exactly the roots of the characteristic polynomial of (22.7). Assuming that these eigenvalues $\lambda_{1,2}$ are distinct, the two corresponding eigenvectors of A are

$$\mathbf{v}^{(1,2)} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix}$$

Recall that if $\lambda_1 \neq \lambda_2$, the general solution of (22.7) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\,,$$

where c_1 and c_2 are arbitrary constants. This function has derivative

$$y'(t) = c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t}\,.$$

Thinking in terms of the corresponding system with $x_1 = y$ and $x_2 = y'$, an equivalent statement is that the general solution of the system (22.8) is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^{(1)} + c_2 e^{\lambda_2 t}\mathbf{v}^{(2)}$$

Is this statement true for all systems of the form $\mathbf{x}' = A\mathbf{x}$? That is, can we take a *linear combination* (or *superposition*) our two particular solutions obtained from the eigenvalues/vectors of A to get the general solution? To answer this question, we'll need some definitions and a theorem.

22.3.1 Linear independence and the Wronskian

Definition. The **Wronskian** of two vector-valued functions $\mathbf{x}^{(1)}(t) = \big(x_{11}(t),\, x_{21}(t)\big)$ and $\mathbf{x}^{(2)}(t) = \big(x_{12}(t),\, x_{22}(t)\big)$ is denoted by $W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)(t)$ and defined as

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t)\,.$$

Note: We often drop the dependence of the Wronksian on the independent variable t, and simply write $W\big(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\big)$. Keep in mind that W is still a function of this independent variable.

Remark. We can write the Wronskian as a matrix determinant:

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = x_{11}x_{22} - x_{12}x_{21} = \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{x}^{(1)} \quad \mathbf{x}^{(2)}$$

Remark. Returning to the connection between 2D systems and 2nd order ODEs, let y_1 and y_2 be two scalar-valued functions (i.e., "normal" functions that output a number) and define $\mathbf{x}^{(1)} = \big(y_1,\, y_1'\big)$ and $\mathbf{x}^{(2)} = \big(y_2,\, y_2'\big)$. Then

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = y_1 y_2' - y_1' y_2 = W(y_1, y_2)\,.$$

Definition. Two vector-valued functions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are **linearly independent** if

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) \neq 0\,.$$

Equivalently, there does not exist a constant α such that $\mathbf{x}^{(1)} = \alpha \mathbf{x}^{(2)}$.

Principle of Superposition. Let $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ be two linearly independent, vector-valued functions that are solutions of the system $\mathbf{x}' = A\mathbf{x}$. Then the general solution of this system is

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t)\,,$$

where c_1 and c_2 are arbitrary constants. We say that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are a **fundamental set of solutions** for $\mathbf{x}' = A\mathbf{x}$.

Example 22.5. Consider the system $\mathbf{x}' = A\mathbf{x}$, where A is as given in Example 22.4:

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}.$$

We have already established that the following functions are two solutions to this system of ODEs:

$$\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\mathbf{v}^{(1)} \quad \text{and} \quad \mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\mathbf{v}^{(2)}\,,$$

where $\lambda_{1,2} = -\dfrac{1}{2} \pm i$ are the eigenvalues of A with corresponding eigenvectors $\mathbf{v}^{(1,2)} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

Do these two solutions make up a fundamental solution set for $\mathbf{x}' = A\mathbf{x}$? That is, are they linearly independent? Let us compute their Wronskian:

$$W\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)(t) = \det\begin{pmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ ie^{\lambda_1 t} & -ie^{\lambda_2 t} \end{pmatrix} = -ie^{(\lambda_1+\lambda_2)t} - ie^{(\lambda_1+\lambda_2)t} = -2ie^{(\lambda_1+\lambda_2)t} \neq 0\,.$$

Therefore, these two functions are a set of fundamental solutions, and the general solution to the given system is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^{(1)} + c_2 e^{\lambda_2 t}\mathbf{v}^{(2)} = c_1 e^{(-\frac{1}{2}+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{(-\frac{1}{2}-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

We could stop here, and say that we have solved this system of equations. However, we note that the solution given above is *complex-valued*, as both the eigenvalues and eigenvectors are complex. Similar to the case of complex roots for 2nd order linear, homogeneous ODEs, we can find *real-valued* fundamental solutions via Euler's formula:

$$e^{it} = \cos(t) + i\sin(t)\,.$$

In particular, it can be shown that the following functions are fundamental solutions of the given system:

$$\mathbf{u}^{(1)}(t) = e^{-t/2}\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}, \qquad \mathbf{u}^{(2)}(t) = e^{-t/2}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix},$$

so that the real-valued general solution is

$$\mathbf{x}(t) = c_1\mathbf{u}^{(1)}(t) + c_2\mathbf{u}^{(2)}(t) = e^{-t/2}\begin{pmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{pmatrix}.$$

In general, the solution to $\mathbf{x}' = A\mathbf{x}$ where A has complex eigenvalues $\lambda_{1,2} = \eta \pm i\mu$ and corresponding eigenvectors $\mathbf{v}^{(1,2)} = \mathbf{a} \pm i\mathbf{b}$ is

$$\boxed{\begin{aligned}&\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)\,,\\ &\mathbf{x}^{(1)}(t) = e^{\eta t}\big(\cos(\mu t)\mathbf{a} - \sin(\mu t)\mathbf{b}\big)\\ &\mathbf{x}^{(2)}(t) = e^{\eta t}\big(\sin(\mu t)\mathbf{a} + \cos(\mu t)\mathbf{b}\big)\end{aligned}}$$

Another way to write this solution is

$$\boxed{\mathbf{x}(t) = e^{\eta t}\big[(c_1\mathbf{a} + c_2\mathbf{b})\cos(\mu t) + (c_2\mathbf{a} - c_1\mathbf{b})\sin(\mu t)\big]}\,.$$

More broadly, we can use the principle of superposition to get the general solution to $\mathbf{x}' = A\mathbf{x}$ for **all the other possible cases** for the eigenvalues/vectors of A:

- $\lambda_{1,2}$ real, $\lambda_1 \neq \lambda_2$. In this case, A has one eigenvector corresponding to each of the eigenvalues, $\mathbf{v}^{1,2}$. These two eigenvectors are linearly independent, so $\mathbf{x}^{(1,2)}(t) = e^{\lambda_{1,2}t}\mathbf{v}^{(1,2}$ is a fundamental solution set, and the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\boxed{\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^{(1)} + c_2 e^{\lambda_2 t}\mathbf{v}^{(2)}}\,.$$

- $\lambda_{1,2}$ real, $\lambda_1 = \lambda_2 =: \lambda$. In this case, there are two possible cases based on whether the single eigenvalue λ has two linearly independent eigenvectors associated with it or not.

  (i) Two linearly independent eigenvectors $\mathbf{v}^{(1,2)}$. In this case, $\mathbf{x}^{(1,2)}(t) = e^{\lambda t}\mathbf{v}^{(1,2)}$ is a fundamental solution set, so the general solution is

$$\boxed{\mathbf{x}(t) = e^{\lambda t}\big(c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)}\big)}\,.$$

  Note: Every such trajectory lies on a straight line through the origin defined by the vector $c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)}$.

  (ii) Only one eigenvector $\mathbf{v}$. In this case, we only have one solution in terms of the eigenvalues/vectors of A: $e^{\lambda t}\mathbf{v}$. In order to build a fundamental set of solutions, we look for something called a **generalized eigenvector** of A: $\mathbf{u}$ such that $A\mathbf{u} = \lambda(\mathbf{u} + \mathbf{v})$. Once we have found this $\mathbf{u}$, we can show that the functions $\mathbf{x}^{(1)}(t) = e^{\lambda t}\mathbf{v}$ and $\mathbf{x}^{(2)} = e^{\lambda t}(\mathbf{u} + t\mathbf{v})$ form a fundamental set of solutions. Then the general solution is:

$$\boxed{\mathbf{x}(t) = e^{\lambda t}\big(c_2 t\mathbf{v} + c_1\mathbf{v} + c_2\mathbf{u}\big)}\,.$$

22.4 Classifying the fixed points of $\mathbf{x}' = A\mathbf{x}$

Let's return to where we began this series of lectures: finding and classifying the fixed points of $\mathbf{x}' = A\mathbf{x}$, starting with our running example.

Example 22.6. Again consider the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}.$$

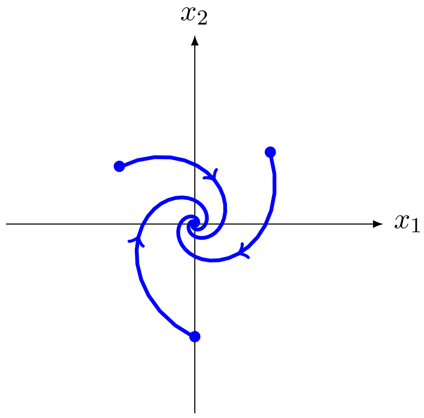
We have shown that the general solution to this system is

$$\mathbf{x}(t) = e^{-t/2}\begin{pmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants. Let's check that $\det A \neq 0$:

$$\det(A) = \frac{1}{4} + 1 = \frac{5}{4} \neq 0,$$

so $\mathbf{x}^* = (0,0)$ is the only fixed point of this system. Looking at each of the components, we see that x_1 and x_2 oscillate with decaying amplitude. Thus, solutions that start "near" $\mathbf{x}^* = (0,0)$ will "spiral in" towards $\mathbf{x}^* = (0,0)$ in the phase plane (oscillations in both components = cycles in phase space):



We call this type of fixed point a **spiral point**. In this example, because the real part of the eigenvalues is negative, the spiral point is **asymptotically stable**.

We can use the general solutions we found previously to classify the stability of the fixed point $\mathbf{x}^* = (0,0)$ of $\mathbf{x}' = A\mathbf{x}$ in all possible cases of the eigenvalues/vectors of A.

(1) $\lambda_1 \neq \lambda_2$, complex ($\lambda_{1,2} = \eta \pm i\mu$). Recall that in this case we have complex conjugate eigenvectors $\mathbf{v}^{(1,2)} = \mathbf{a} \pm i\mathbf{b}$, and the general solution reads:

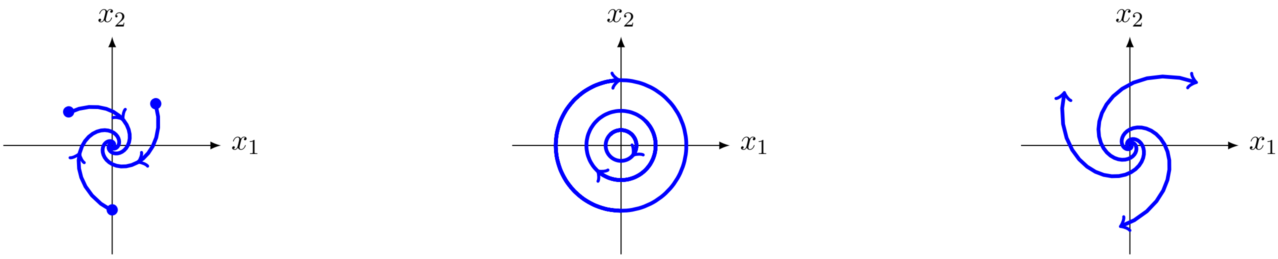
$$\mathbf{x}(t) = e^{\eta t}\left[(c_1\mathbf{a} + c_2\mathbf{b})\cos(\mu t) + (c_2\mathbf{a} - c_1\mathbf{b})\sin(\mu t)\right]$$

Based on the form of this general solution, we see that its general behavior is oscillations (cycles) in phase space with changing amplitude according to the term $e^{\eta t}$ (the direction of the oscillations - clockwise or counterclockwise - depends on the constants c_1 and c_2 and the vectors $\mathbf{a}$ and $\mathbf{b}$). Thus, there are three possible cases depending on the sign of η:

(i) $\eta < 0$. Solutions oscillate with exponentially decaying amplitude; this is the case we just saw in our example. We call $(0,0)$ an **asymptotically stable spiral point**.

(ii) $\eta = 0$. Solutions oscillate with constant amplitude. We call $(0,0)$ a **center**; this fixed point is **stable**, but not asymptotically stable.

(iii) $\eta > 0$. Solutions oscillate with exponentially growing amplitude. We call $(0,0)$ an **unstable spiral point**.



(2) $\lambda_1 \neq \lambda_2$, real. Recall that in this case, the general solution is

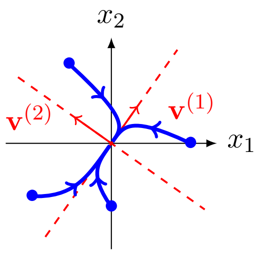
$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^{(1)} + c_2 e^{\lambda_2 t}\mathbf{v}^{(2)} ,$$

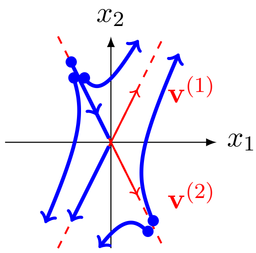
where $\mathbf{v}^{(1,2)}$ are the eigenvectors corresponding to $\lambda_{1,2}$. Based on the form of this general solution, we see that its general behavior is some mixture of either exponential growth or decay based on the sign of each eigenvalue λ_1 and λ_2. Thus, there are three possible cases:

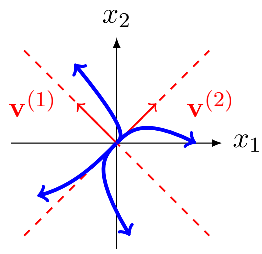
(i) $\lambda_1 < 0,\ \lambda_2 < 0$. Solutions exponentially decay towards zero in both components. We call $(0,0)$ an **asymptotically stable node**.

(ii) $\lambda_1 > 0,\ \lambda_2 < 0$. Some solutions may initially exponentially decay due to the $e^{\lambda_2 t}\mathbf{v}^{(2)}$ term, but all solutions eventually move away from the origin according to the $e^{\lambda_1}\mathbf{v}^{(1)}$ term. We call $(0,0)$ a **saddle point**; this fixed point is **unstable**. Solutions with initial conditions on the line defined by $\mathbf{v}^{(2)}$ are characterized by $c_1 = 0$; thus, they stay on this line for all t and tend toward $(0,0)$. We call the line defined by $\mathbf{v}^{(2)}$ (i.e., the eigenvector corresponding to the negative eigenvalue) the **stable manifold**.

(iii) $\lambda_1 > 0,\ \lambda_2 > 0$. Solutions tend away from the origin exponentially (i.e., exponential growth in magnitude, components may be negative or positive). We call $(0,0)$ an **unstable node**.





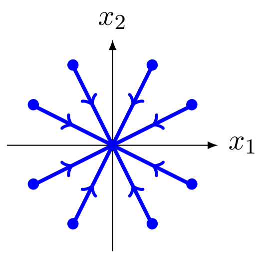


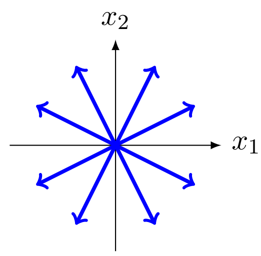
(3) $\lambda_1 = \lambda_2 =: \lambda$ (real). Recall that there were two sub-cases for the case of repeated eigenvalues depending on the number of linearly independent eigenvectors:

(i) Two linearly independent eigenvectors $\mathbf{v}^{(1,2)}$. In this case, the general solution is

$$\mathbf{x}(t) = e^{\lambda t}\big(c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)}\big) ,$$

and we call $(0,0)$ a **proper node**. Recall that such solutions lie on straight lines through the origin: $c_1\mathbf{v}^{(1)} + c_2\mathbf{v}^{(2)}$. If $\lambda < 0$, these solutions move towards the origin on these lines as t increases, and if $\lambda > 0$, they move away.



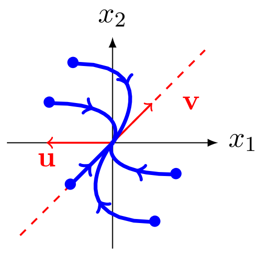


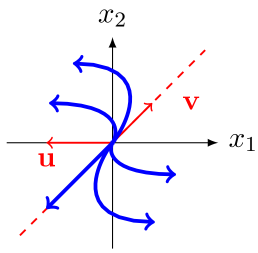
(ii) Only one eigenvector $\mathbf{v}$. In this case, the general solution is

$$\mathbf{x}(t) = e^{\lambda t}\big(c_2 t\mathbf{v} + c_1\mathbf{v} + c_2\mathbf{u}\big) ,$$

where $\mathbf{u}$ is a generalized eigenvector of A: $A\mathbf{u} = \lambda(\mathbf{u} + \mathbf{v})$. In this case, we call $(0,0)$ an **improper node**. We note that the magnitude of these solutions is given by $e^{\lambda t}$, while their direction (i.e., position in phase space) is given by the vector expression $c_2 t\mathbf{v} + c_1\mathbf{v} + c_2\mathbf{u}$. (For fixed c_1 and c_2, this vector expression is exactly the line parallel to $\mathbf{v}$ that passes through the point $c_1\mathbf{v} + c_2\mathbf{u}$ at $t = 0$. For a discussion of how to obtain the trajectories of this system from this expression, see Boyce & DiPrima Chapter 9.1)

Thus, if $\lambda < 0$, these solutions move towards the origin as t increases, and if $\lambda > 0$, they move away.





In either of the cases (3.1) and (3.2), if:

- $\lambda < 0$, $(0,0)$ is **asymptotically stable**.
- $\lambda > 0$, $(0,0)$ is **unstable**.

Eigenvalues	**Type**	**Stability**
$\lambda_1 > \lambda_2 > 0$	node	unstable
$\lambda_2 < \lambda_1 < 0$	node	asymptotically stable
$\lambda_1 > 0 > \lambda_2$	saddle point	unstable
$\lambda_1 = \lambda_2 > 0$	proper or improper node	unstable
$\lambda_1 = \lambda_2 < 0$	proper or improper node	asymptotically stable
$\lambda_{1,2} = \eta \pm i\mu$ - - - - - - - - - - - - $\eta > 0$	spiral point	unstable
$\eta = 0$	center	stable
$\eta < 0$	spiral point	asymptotically stable

Table 22.1: Stability of $\mathbf{x}^* = (0,0)$ based on the eigenvalues of A in $\mathbf{x}' = A\mathbf{x}$.

Lecture 24: Numerical Phase Portrait (Python Demo)

The Python code (in the form of Jupyter notebooks) used for this demonstration can be **found in this GitHub repository under the folder `phase_plane`**. The following notes on linear stability analysis can be used as a supplement to this demonstration, as this topic is not covered in the preceding course notes.

Linear Stability Analysis of Non-linear Systems

Let us consider a *non-linear* 2D autonomous system:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where the vector-valued function $\mathbf{f}$ is non-linear. That is to say, one or both of its component functions f_1 and/or f_2 are non-linear functions. We saw an example of a non-linear autonomous system in Lectures 20-21:

$$\begin{cases} x_1' = -(x_1 - x_2)(1 - x_1 - x_2) \\ x_2' = x_1(2 + x_2) \end{cases}$$

In those lectures, we saw that this system had multiple fixed points, and that we could qualitatively describe how solutions behaved near these fixed points.

In the past two lectures, we learned how to classify the fixed points of a *linear* autonomous system of the form $\mathbf{x}' = A\mathbf{x}$ based on the *eigenvalues* of the matrix A, as these told us how solutions would behave near the only fixed point of this system, $\mathbf{x}^* = \mathbf{0}$. Can we do something similar to classify the fixed points of a more general non-linear autonomous system?

If f_1 and f_2 have continuous first and second partial derivatives, then we can use a *Taylor expansion* about the point $\mathbf{x} = \mathbf{x}^*$ to approximate $f(\mathbf{x})$ by a linear function near the point $\mathbf{x}^*$:

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}^*) + \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}^*) & \dfrac{\partial f_1}{\partial x_2}(\mathbf{x}^*) \\ \dfrac{\partial f_2}{\partial x_1}(\mathbf{x}^*) & \dfrac{\partial f_2}{\partial x_2}(\mathbf{x}^*) \end{pmatrix} \big(\mathbf{x} - \mathbf{x}^*\big)$$

The matrix of partial derivatives of $\mathbf{f}$ is called the **Jacobian matrix** of the function $\mathbf{f}$, denoted $J(\mathbf{x})$. Using the fact that $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, this approximation becomes:

$$\mathbf{x}' \approx J(\mathbf{x}^*)\big(\mathbf{x} - \mathbf{x}^*\big). \tag{1}$$

We note that this approximation is only "good" (in some sense of accuracy) when $\mathbf{x}$ is "close to" $\mathbf{x}^*$. Now consider the change of coordinates $\mathbf{u} = \mathbf{x} - \mathbf{x}^*$. Then $\mathbf{u}' = \mathbf{x}'$, and the approximation in (1) reads:

$$\mathbf{u}' = J(\mathbf{x}^*)\mathbf{u}.$$

Thus, the dynamics of $\mathbf{u} = \mathbf{x} - \mathbf{x}^*$ are approximately that of a linear homogeneous system with constant coefficients near the point $\mathbf{u} = \mathbf{0}$ (a.k.a. $\mathbf{x} = \mathbf{x}^*$). That is to say, we can approximate the trajectories of this system with those of the corresponding linear homogeneous system with constant coefficients near the fixed point. In particular, we can use the eigenvalues of $J(\mathbf{x}^*)$ to determine the stability of $\mathbf{u} = \mathbf{0}$, and thus, the stability of $\mathbf{x} = \mathbf{x}^*$. This whole process of classifying fixed points of a non-linear system via the Jacobian matrix is called **linear stability analysis**.

The following table summarizes all possible cases of the stability of a fixed point $\mathbf{x}^*$ based on the eigenvalues of the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}^*$, $J(\mathbf{x}^*)$. We note that there are some differences in how we can classify fixed points of non-linear systems versus linear systems. These differences are due to the fact that this analysis comes from *approximating* $\mathbf{f}(\mathbf{x})$ with a linear function near $\mathbf{x} = \mathbf{x}^*$. Thus, we would expect the solutions of the nonlinear system to not behave exactly like those of a linear system, especially as they get further from the fixed point.

Eigenvalues	Type	Stability
$\lambda_1 > \lambda_2 > 0$	node	unstable
$\lambda_2 < \lambda_1 < 0$	node	asymptotically stable
$\lambda_1 > 0 > \lambda_2$	saddle point	unstable
$\lambda_1 = \lambda_2 > 0$	*node* or *spiral point*	unstable
$\lambda_1 = \lambda_2 < 0$	*node* or *spiral point*	asymptotically stable
$\lambda_{1,2} = \eta \pm i\mu$ $\eta > 0$	spiral point	unstable
$\eta = 0$	center *or spiral point*	*indeterminate*
$\eta < 0$	spiral point	asymptotically stable

Stability of a fixed point $\mathbf{x}^*$ of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ based on the eigenvalues of the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x} = \mathbf{x}^*$. Differences between non-linear and linear systems have been highlighted in *italics*.